

Asymptotics of one dimensional forest fire process with non instantaneous propagation

Jean-Maxime Le Cousin

March 13, 2015

Abstract

Consider the following forest fire model where the possible locations of trees are the sites of \mathbb{Z} . Each site has three possible states: 'vacant', 'occupied' or 'burning'. Vacant sites become occupied at rate 1. At each site, ignition (by lightning) occurs at rate λ . When a site is ignited, a fire starts and propagates to neighbors at rate π . We study the asymptotic behavior of this process as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$. We show that there are three possible classes of scaling limits, according to the regime in which $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$.

Contents

1	Introduction	3
1.1	The discrete model	3
1.2	Motivation and references	3
1.3	Plan of the paper	6
2	Main results	6
2.1	Notation	6
2.2	Heuristic scales and relevant quantities	7
2.3	Main results when $p \in [0, \infty)$	9
2.4	Main results for $p = \infty$	13
3	Existence and uniqueness of the limit process	16
3.1	Restriction of the LFFP(∞, z_0) to a finite box	16
3.2	Restriction of the LFFP(p) to a finite box	17
4	Propagation Lemmas	21
4.1	Propagation lemma in the regime $\mathcal{R}(p)$, for some $p \in (0, \infty)$	22
4.2	Propagation lemma in the regime $\mathcal{R}(0)$	25
4.3	Propagation lemma in the regime $\mathcal{R}(\infty, z_0)$	26
4.4	Application to the (λ, π) -FFP	27
5	Localization of the (λ, π)-FFP	32
6	Localization of the result	43
6.1	Localization in the regime $\mathcal{R}(p)$	43
6.2	Localization in the regime $\mathcal{R}(\infty, z_0)$	44
7	Convergence in the regime $\mathcal{R}(\infty, z_0)$	45
7.1	Occupation of vacant zone	45
7.2	Height of the barrier	46
7.3	Proof of Theorem 6.2	48

8	Convergence in the regime $\mathcal{R}(p)$, for some $p > 0$	53
8.1	Occupation of vacant zone	54
8.2	Height of the barrier	55
8.3	Persistent effect of microscopic fires	57
8.4	Heart of the proof	61
8.5	Proof of Theorem 6.1 for $p > 0$	85
8.6	Cluster size distribution when $p > 0$	89
9	Convergence in the regime $\mathcal{R}(0)$	93
9.1	Occupation of vacant zone	94
9.2	Height of the barrier	94
9.3	Persistent effect of microscopic fires	97
9.4	Heart of the proof	99
9.5	Proof of Theorem 6.1 for $p = 0$	107
9.6	Cluster size distribution when $p = 0$	110

1 Introduction

This section is devoted to preliminaries. We first define the (λ, π) -forest fire process with non instantaneous propagation. We then recall some known results about forest fire processes. Finally, we give the plan of the present paper.

1.1 The discrete model

Here we introduce the forest fire model with non instantaneous propagation.

Definition 1.1. Let $\lambda \in (0, 1]$ and $\pi \geq 1$ be fixed. For each $i \in \mathbb{Z}$, we consider three Poisson processes, $N^S(i) = (N_t^S(i))_{t \geq 0}$, $N^M(i) = (N_t^M(i))_{t \geq 0}$ and $N^P(i) = (N_t^P(i))_{t \geq 0}$ with respective parameters 1, λ and π , all of these processes being independent. Consider a $\{0, 1, 2\}$ -valued process $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ such that a.s., for all $i \in \mathbb{Z}$, $(\eta_t^{\lambda, \pi}(i))_{t \geq 0}$ is càdlàg. We say that $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ is a (λ, π) -forest fire process ((λ, π) -FFP in short) if a.s., for all $i \in \mathbb{Z}$, all $t \geq 0$,

$$\begin{aligned} \eta_t^{\lambda, \pi}(i) = & \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi}(i)=0\}} dN_s^S(i) + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi}(i)=1\}} dN_s^M(i) \\ & + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi}(i+1)=2, \eta_{s-}^{\lambda, \pi}(i)=1\}} dN_s^P(i+1) + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi}(i-1)=2, \eta_{s-}^{\lambda, \pi}(i)=1\}} dN_s^P(i-1) \\ & - 2 \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi}(i)=2\}} dN_s^P(i). \end{aligned}$$

Formally, we say that $\eta_t^{\lambda, \pi}(i) = 0$ if there is no tree at site i at time t and $\eta_t^{\lambda, \pi}(i) = 1$ if the site i is occupied. The case $\eta_t^{\lambda, \pi}(i) = 2$ means that the site i is burning. Thus, the forest fire process starts from an empty initial configuration, seeds fall according to some i.i.d. Poisson processes of parameter 1 and matches fall according to some i.i.d. Poisson processes of parameter λ . When a seed falls on an empty site, a tree appears immediately. When a match falls on an occupied site, a fire starts and waits for an exponential time of parameter π before it propagates to its neighbors and vanishes. If its right (resp. left) neighbor is occupied then it becomes burning. Seeds falling on occupied sites, matches falling on vacant sites and fires propagating to vacant sites have no effect.

This process can be shown to exist and to be unique (for almost every realization of N^S, N^M, N^P) by using a *graphical construction*. Indeed, to build the process until a given time $T > 0$, it suffices to work between sites i which are vacant until time T [because $N_T^S(i) = 0$]. Interaction cannot cross such sites. Since such sites are a.s. infinitely many, this allows us to handle a graphical construction. It should be pointed out that this construction only works in dimension 1.

For $a, b \in \mathbb{Z}$, we set $\llbracket a, b \rrbracket = \{a, \dots, b\} \subset \mathbb{Z}$. For $\eta \in \{0, 1, 2\}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we define the occupied connected component around i as

$$C(\eta, i) = \begin{cases} \emptyset & \text{if } \eta(i) = 0 \text{ or } 2, \\ \llbracket l(\eta, i), r(\eta, i) \rrbracket & \text{if } \eta(i) = 1, \end{cases}$$

where $l(\eta, i) = \sup\{k < i : \eta(k) = 0 \text{ or } 2\} + 1$ and $r(\eta, i) = \inf\{k > i : \eta(k) = 0 \text{ or } 2\} - 1$.

1.2 Motivation and references

Consider a graph $G = (S, A)$, S being the set of vertices and A the set of edges. Introduce the space of configurations $E = \{0, 1, 2\}^S$. For $\eta \in E$, we say that $\eta(i) = 0$ if the site $i \in S$ is vacant, $\eta(i) = 1$ if the site i is occupied by a tree and $\eta(i) = 2$ if the tree in i is burning. Two sites are neighbors if there is an edge between them. We call forests the connected components of occupied sites. For $i \in S$ and $\eta \in E$, we denote by $C(\eta, i)$ the forest around i in the configuration η (with $C(\eta, i) = \emptyset$ if $\eta(i) = 0$ or $\eta(i) = 2$). We consider the following rules

- vacant sites become occupied (a seed falls and a tree immediately grows) at rate 1;
- occupied sites take fire (a match falls) at rate $\lambda > 0$;

- fires propagate to neighbors (inside the forest) at rate $\pi > 0$.

Such a model was introduced by Henley [14] and Drossel and Schwabl [9] as a toy model for forest fire propagation and as an example of a simple model intended to clarify the concept of *self-organized criticality*.

The study of self-organized critical systems has become rather popular in physics since the end of the 80's. These are simple models supposed to clarify temporal and spatial randomness observed in a variety of natural phenomena showing long range correlations, like sand piles, avalanches, earthquakes, stock market crashes, forest fires, shapes of mountains, clouds, etc. It is remarkable that such phenomena, reminiscent of critical behavior, arise so frequently in nature where nobody is here to finely tune the parameters to critical values. The most classical model is the sand pile model introduced in 1987 in [1], but many variants or related models have been proposed and studied more or less rigorously, describing earthquakes (see [16]) or forest fires (see [14]).

The features of the model depend on the geometry of the graph; we only consider in this paper the case $S = \mathbb{Z}$ (with its natural set of edges). They also depend on the laws of the processes governing seeds, matches and propagation. We work here in the classical case where all processes are Poisson processes.

From the point of view of self-organized criticality, the interesting regime is the asymptotic behavior of the forest-fire process as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$: then fires are very rare, but concern huge occupied components. We present three possible limit processes (depending on the regime at which $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$) arising when we suitably rescale space and accelerate time.

Forest fire on \mathbb{Z}

All the available results concern the limit case where the propagation is instantaneous ($\pi = \infty$): when a tree takes fire, the whole forest (to which it belongs) is *destroyed immediately*. The model is thus:

- vacant sites become occupied (a seed falls and a tree immediately grows) at rate 1;
- matches fall on occupied sites at rate λ and then burn instantaneously the corresponding forest.

We denote $\eta_t^\lambda \in \{0, 1\}^{\mathbb{Z}}$ the configuration at time t . Observe that (possible) infinite clusters in the initial configuration would immediately disappear.

The following results are related to this model.

Asymptotic density

Van den Berg and Járai study in [4] the asymptotic density of vacant sites in the limit $\lambda \rightarrow 0$. Their result states that there are two constants $0 < c < C$ such that for any initial configuration, for any $\lambda > 0$ small enough, for t large enough (of order $\log(1/\lambda)$),

$$\frac{c}{\log(1/\lambda)} \leq \mathbb{P} [\eta_t^\lambda(0) = 0] \leq \frac{C}{\log(1/\lambda)}.$$

This is coherent with the intuition that the rarer fires are, the more space is occupied by trees (although because of the lack of monotonicity, this is not straightforward). We mention that such a result was stated in Drossel-Clar-Schwabl [8]. But the proof in [8] is not rigorous: it is based on the *ansatz* that the cluster sizes were following a cutoff power law, for cluster-sizes up to some s_{max}^λ defined by $s_{max}^\lambda \log s_{max}^\lambda = 1/\lambda$, i.e.

$$s_{max}^\lambda \simeq \frac{1}{\lambda \log(1/\lambda)}.$$

In [4], van den Berg and Járai also show that the cluster sizes cannot follow the predicted power law.

Sizes of clusters, first results

In [7], Brouwer and Pennanen show that this last *ansatz* holds true up to $s_{max}^{1/3}$. More specifically, they show that there are some constants $0 < c < C$ such that for all $0 < \lambda < 1$ and all stationary measures μ_λ (invariant by translation) of the forest fire model on \mathbb{Z} with parameter λ , for all $x < (s_{max}^\lambda)^{1/3}$,

$$\frac{c}{(1+x)\log(1/\lambda)} \leq \mu_\lambda(|C(\eta, 0)| = x) \leq \frac{C}{(1+x)\log(1/\lambda)}.$$

Observe that this estimate is valid for relatively small clusters that will not be seen after rescaling (microscopic clusters).

Scaling limits

Still in the limit case where the propagation is instantaneous, Bressaud and Fournier have proved in [5] that in the asymptotic of rare matches, the forest fire process converges, under suitable normalization, to some limit forest fire process. They described precisely the dynamics of this limit process and have shown that it is unique, that it can be built by using a graphical construction and thus can be perfectly simulated. Using the limit process, they have also estimated the size of clusters. Very roughly, they have proved that in a very weak sense, for λ small enough and for t large enough (of order $\log(1/\lambda)$), the cluster-size distribution resembles

$$\mathbb{P}[C(\eta_t^\lambda, 0) = x] \simeq \frac{a}{(x+1)\log(1/\lambda)} \mathbf{1}_{\{x \ll 1/(\lambda \log(1/\lambda))\}} + b\lambda \log(1/\lambda) e^{-x\lambda \log(1/\lambda)},$$

where a, b are two positive constants. This means that there are two types of clusters: *microscopic clusters*, described by a power-like law and *macroscopic clusters*, described by an exponential-like law. This shows a *phase transition* around the *critical size* $1/(\lambda \log(1/\lambda))$.

In [6], Bressaud and Fournier have extended their results by replacing Poisson processes by the case where seeds (respectively matches) fall on each site of \mathbb{Z} independently, according to some stationary renewal processes, with stationary delay distributed according to some law ν_S (respectively ν_M^λ). This means that for any time $t \geq 0$ and on any site $i \in \mathbb{Z}$, the time we have to wait for the next seed is a ν_S -distributed random variable. They also assume that ν_S has a bounded support or a tail with fast or regular or slow variations. They prove that, after rescaling, the corresponding forest fire process converges, as $\lambda \rightarrow 0$, to a limit process. They show that there are four classes of limit processes, according to the fact that

- ν_S has a bounded support,
- ν_S has a tail with fast decay,
- ν_S has a tail with polynomial decay,
- ν_S has a tail with logarithmic decay.

They see that the limit forest fire process build in [5] is quite universal: it describes the asymptotics of a large class (roughly exponential decay for ν_S) of forest fire processes. A similar limit process arises when ν_S has bounded support. But some quite different limit processes arise when ν_S has a heavy tail.

Main idea of the present paper

From the modelling point of view, the instantaneously destroying of clusters is not clearly justified. The goal of this paper is to extend the result in [5] to the case where fires need a random time to propagate to neighbors.

We thus consider the case where seeds (resp. matches) fall on each site of \mathbb{Z} independently, according to some Poisson processes with parameter 1 (resp. λ) and where a burning tree has to wait for an exponential time of parameter π to propagate to neighbors. Since the scaling in [5] depends only on the seed and match processes (i.e. only on 1 and λ), the time and space scales will be the same here. We will separate three cases :

- the case where fires propagate very fast;
- the case where fires propagate very slowly;
- the intermediate case.

The first case is the most physically realistic and the most widely used. We will show that, if π is large, then everything happens as if $\pi = \infty$ (instantaneous propagation). The other cases are even though mathematically interesting.

1.3 Plan of the paper

In Section 2, we start by explaining the heuristic scales and the relevant quantities (rescaled macroscopic clusters and measure of microscopic clusters). We then give our main results (scaling limits and cluster-size distribution) together with heuristic proof. In Section 3, we study the existence and uniqueness of the limit process. In Section 4, we study the effect of fires in the discrete process, which will be usefull in the rest of the paper (propagation through an occupied zone). In Section 5, we give a discrete version of Section 3. The rest of the paper is devoted to the rigorous proof of our results: we treat the convergence in the regime $\mathcal{R}(\infty, z_0)$ in Section 7, in the regime $\mathcal{R}(p)$, for some $p \in (0, \infty)$ in Section 8 and finally in the regime $\mathcal{R}(0)$ in Section 9. In the end of each two last sections, we deduce estimates on the cluster size distribution for the process.

2 Main results

2.1 Notation

In the whole paper, we use the convention $1/\infty = 0$ and $1/0 = \infty$.

We denote, for $J = [a, b]$ an interval of \mathbb{R} , by $|J| = b - a$ the length of J and for $\alpha > 0$, we set $\alpha J = [\alpha a, \alpha b]$.

For $I \subset \mathbb{Z}$, $|I| = \#I$ stands for the number of elements in I . For $I = \llbracket a, b \rrbracket = \{a, \dots, b\} \subset \mathbb{Z}$ and $\alpha > 0$, we will set $\alpha I := [\alpha a, \alpha b] \subset \mathbb{R}$. For $\alpha > 0$, we of course take the convention that $\alpha \emptyset = \emptyset$.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the integer part of x .

We denote by $\mathcal{I} = \{[a, b], a \leq b\}$ the set of all closed finite intervals of \mathbb{R} . For two intervals $[a, b]$ and $[c, d]$, we set

$$\delta([a, b], [c, d]) = |a - c| + |b - d|, \quad \delta([a, b], \emptyset) = |b - a|.$$

For $(x, I), (y, J)$ in $\mathbb{D}([0, T], \mathbb{R}_+ \times \mathcal{I} \cup \{\emptyset\})$, the set of càdlàg functions from $[0, T]$ into $\mathbb{R}_+ \times \mathcal{I} \cup \{\emptyset\}$, we define

$$\mathbf{d}_T((x, I), (y, J)) = \int_0^T \left[|x(t) - y(t)| + \delta(I_t, J_t) \right] dt.$$

For two functions $I, J: [0, T] \rightarrow \mathcal{I} \cup \{\emptyset\}$, we set

$$\delta_T(I, J) = \int_0^T \delta(I_t, J_t) dt.$$

For $(x, t) \in \mathbb{R} \times [0, T]$ we also set, for $p \geq 0$,

$$\Lambda_{(x,t)}^p := \{(x + z, t - p|z|) : |z| \leq t/p\}$$

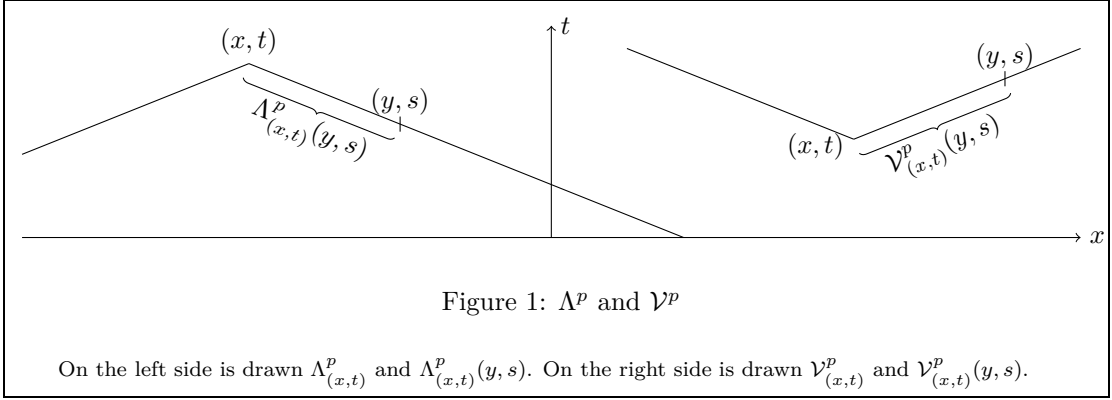
$((r, v) \in \Lambda_{(x,t)}^p) \iff v = t - p|r - x|$ and its part which joins (y, s) to (x, t)

$$\Lambda_{(x,t)}^p(y, s) = \begin{cases} \{(z, t - p|z - x|) : z \in [x, y]\} & \text{if } (y, s) \in \Lambda_{(x,s)}^p \text{ and } y > x, \\ \{(z, t - p|z - x|) : z \in [y, x]\} & \text{if } (y, s) \in \Lambda_{(x,s)}^p \text{ and } y < x, \\ \emptyset & \text{else.} \end{cases}$$

Similarly, we define

$$\begin{aligned} \mathcal{V}_{(x,t)}^p &= \{(x+z, t+p|z|) : z \in \mathbb{R}\} \\ \mathcal{V}_{(x,t)}^p(y, s) &= \begin{cases} \{(z, t+p|z-x|) : z \in [x, y]\} & \text{if } (y, s) \in \mathcal{V}_{(x,t)}^p \text{ and } y > x, \\ \{(z, t+p|z-x|) : z \in [y, x]\} & \text{if } (y, s) \in \mathcal{V}_{(x,t)}^p \text{ and } y < x, \\ \emptyset & \text{else,} \end{cases} \end{aligned}$$

see Figure 1. Observe that $\Lambda_{(x,t)}^p(y, s) = \mathcal{V}_{(y,s)}^p(x, t)$. Also observe that $\Lambda_{(x,t)}^0 = \mathcal{V}_{(x,t)}^0 = \{(z, t) : z \in \mathbb{R}\}$.



2.2 Heuristic scales and relevant quantities

We look for some time scale for which tree clusters see about one fire per unit of time. But for λ very small, clusters will be very large before a match falls inside. We thus also have to rescale space. Since this does not depend on π , these scales are the same as in [5]. We also have to find the different regimes at which $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$.

Time scale

For $\lambda > 0$ very small and for t not too large, one might neglect fires, so that roughly, each site is vacant with probability e^{-t} . Indeed, the time we have to wait for the first seed follows, on each site, the law $\mathcal{E}(1)$. Thus $C(\eta_t^{\lambda, \pi}, 0) \simeq \llbracket -X, Y \rrbracket$, where X, Y are geometric random variables with parameter e^{-t} . Consequently, for t not too large,

$$|C(\eta_t^{\lambda, \pi}, 0)| \simeq e^t.$$

On the other hand, the rate at which matches fall in the cluster $C(\eta_t^{\lambda, \pi}, 0)$ is $\lambda|C(\eta_t^{\lambda, \pi}, 0)|$. So we decide to accelerate time by a factor

$$\mathbf{a}_\lambda = \log(1/\lambda). \quad (2.1)$$

In this way, $\lambda|C(\eta_{\mathbf{a}_\lambda}^{\lambda, \pi}, 0)| \simeq 1$.

Space scale

We now rescale space in such a way that during a time interval of order $\mathbf{a}_\lambda = \log(1/\lambda)$, something like one match falls per unit of (space) length. Since fires occur at rate λ , our space scale has to be of order

$$\mathbf{n}_\lambda = \left\lfloor \frac{1}{\lambda \mathbf{a}_\lambda} \right\rfloor = \left\lfloor \frac{1}{\lambda \log(1/\lambda)} \right\rfloor. \quad (2.2)$$

This means that we will identify $\llbracket 0, \mathbf{n}_\lambda \rrbracket \subset \mathbb{Z}$ with $[0, 1] \subset \mathbb{R}$.

Propagation velocity

The time needed for a fire to destroy a macroscopic cluster (which contains about \mathbf{n}_λ sites) is of order $\frac{\mathbf{n}_\lambda}{\pi}$. Indeed, a burning tree waits for an exponential time of parameter π before it propagates to neighbors. Thus, if a fire starts at 0, it needs roughly a time \mathbf{n}_λ/π to reach \mathbf{n}_λ . We have to compare the time \mathbf{n}_λ/π to the characteristic time \mathbf{a}_λ . Thus we have to separate the three following regimes, as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ (observe that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \simeq \frac{1}{\lambda \log^2(1/\lambda) \pi}$):

- $\frac{1}{\lambda \log^2(1/\lambda) \pi} \rightarrow 0$, which corresponds to the case where fires propagate very fast;
- $\frac{1}{\lambda \log^2(1/\lambda) \pi} \rightarrow p$, for some $p \in (0, \infty)$, which is an intermediate case;
- $\frac{1}{\lambda \log^2(1/\lambda) \pi} \rightarrow \infty$, which corresponds to the case where fires propagate very slowly.

Recall that, when neglecting fires and for $t < 1$, $1/\lambda^t$ is the order of magnitude of the occupied cluster around 0 at time $\mathbf{a}_\lambda t$. Thus a match falling in 0 at time $\mathbf{a}_\lambda t$ needs a time of order $1/(\lambda^t \pi)$ to destroy the whole component. In order to treat the last case, we suppose that there exists $z_0 \in [0, 1)$ such that

$$\frac{1}{\lambda^t \pi} \rightarrow \begin{cases} 0 & \text{if } t < z_0, \\ \infty & \text{if } t > z_0. \end{cases} \quad (2.3)$$

This means that if the match falls at time $\mathbf{a}_\lambda t < \mathbf{a}_\lambda z_0$, there are few occupied sites around 0. Thus the fire destroys the whole component in a time of order $1/(\lambda^t \pi) \ll \mathbf{a}_\lambda$. On the other hand, if the match falls a time $\mathbf{a}_\lambda t > \mathbf{a}_\lambda z_0$ then the component is too big to be destroyed before $\mathbf{a}_\lambda T$, for all $T > 0$.

To summarize, we will treat separately the three following regimes, as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$.

1. $\mathcal{R}(0)$: $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \ll 1$, the fast regime;
2. $\mathcal{R}(p)$: $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \sim p \in (0, \infty)$, the intermediate regime;
3. $\mathcal{R}(\infty, z_0)$: $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \gg 1$ and $\frac{\log(\pi)}{\log(1/\lambda)} \rightarrow z_0 \in [0, 1]$, the slow regime.

Rescaled clusters

We thus set, for $\lambda \in (0, 1)$, $\pi \geq 1$, $t \geq 0$ and $x \in \mathbb{R}$, recalling Subsection 2.1,

$$D_t^{\lambda, \pi}(x) := \frac{1}{\mathbf{n}_\lambda} C\left(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x \rfloor\right). \quad (2.4)$$

However, this creates an immediate difficulty: recalling that $C(\eta_t^{\lambda, \pi}, 0) \simeq e^t$ for t not too large, we see that for each site x , $|D_t^{\lambda, \pi}(x)| \simeq \lambda \log(1/\lambda) e^{t \log(1/\lambda)} = \lambda^{1-t} \log(1/\lambda)$, of which the limit when $\lambda \rightarrow 0$ is 0 for $t < 1$ and $+\infty$ for $t \geq 1$.

For $t \geq 1$, there might be fires in effect and one hopes that this will make the possible limit of $|D_t^{\lambda, \pi}(x)|$ finite. However, fires can only reduce the size of clusters so that for $t < 1$, the limit of $|D_t^{\lambda, \pi}(x)|$ will really be 0. This cannot be a Markov process because it remains at 0 during a time interval of length exactly 1. We thus need to keep track of more information in order to control when it exits from 0.

To have an idea of the sizes of microscopic clusters, we keep some information about *the degree of smallness* of microscopic clusters. We consider

$$\mathbf{m}_\lambda = \left\lfloor \frac{1}{\lambda \mathbf{a}_\lambda^2} \right\rfloor = \left\lfloor \frac{1}{\lambda \log^2(1/\lambda)} \right\rfloor. \quad (2.5)$$

Remark that $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$ but $\mathbf{m}_\lambda \gg \lambda^{-t}$, for all $t \in [0, 1)$. We introduce, for $\lambda > 0$, $\pi \geq 1$, $x \in \mathbb{R}$, $t \geq 0$,

$$K_t^{\lambda, \pi}(x) = \frac{\left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda \rrbracket : \eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 1 \right\} \right|}{2\mathbf{m}_\lambda + 1} \in [0, 1], \quad (2.6)$$

$$Z_t^{\lambda, \pi}(x) = \frac{-\log(1 - K_t^{\lambda, \pi}(x))}{\log(1/\lambda)} \wedge 1 \in [0, 1]. \quad (2.7)$$

Observe that $K_t^{\lambda,\pi}(x)$ stands for the *local density of occupied sites* around $\lfloor \mathbf{n}_\lambda x \rfloor$ at time $\mathbf{a}_\lambda t$. This density is *local* because $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$. We hope that for $t < 1$, neglecting fires, $K_t^{\lambda,\pi}(x) \simeq 1 - \lambda^t$, whence $Z_t^{\lambda,\pi}(x) \simeq t$.

For all $\lambda > 0$ small enough (we need that $2\mathbf{m}_\lambda + 1 < 1/\lambda$), it also holds that $Z_t^{\lambda,\pi}(x) = 1$ if and only if $K_t^{\lambda,\pi}(x) = 1$, i.e. if and only if all the sites are occupied around $\lfloor \mathbf{n}_\lambda x \rfloor$. Indeed, $Z_t^{\lambda,\pi}(x) = 1$ implies that $-\log(1 - K_t^{\lambda,\pi}(x)) \geq \log(1/\lambda)$, so that $K_t^{\lambda,\pi}(x) \geq 1 - \lambda > 1 - 1/(2\mathbf{m}_\lambda + 1)$, whence $K_t^{\lambda,\pi}(x) = 1$.

Final description

We will study the (λ, π) -FFP through $(D_t^{\lambda,\pi}(x), Z_t^{\lambda,\pi}(x))_{t \geq 0, x \in \mathbb{R}}$. The main idea is that for $\lambda > 0$ very small and $\pi \geq 1$ large enough:

- if $Z_t^{\lambda,\pi}(x) = z \in (0, 1)$, then $|D_t^{\lambda,\pi}(x)| \simeq 0$ and the (rescaled) cluster containing x is microscopic (in the sense that the non-rescaled cluster containing $\lfloor \mathbf{n}_\lambda x \rfloor$ is small when compared to \mathbf{n}_λ), but we control the local density of occupied sites around x , which resembles $1 - \lambda^z$. Observe that this density tends to 1 as $\lambda \rightarrow 0$ for all $z \in (0, 1)$;
- if $Z_t^{\lambda,\pi}(x) = 1$ and $D_t^{\lambda,\pi}(x) = [a, b]$, then the (rescaled) cluster containing x is macroscopic and has a length equal to $|b - a|$ (or $|C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}, \lfloor \mathbf{n}_\lambda x \rfloor)| \simeq \mathbf{n}_\lambda |b - a|$ in the original scales).

Definition 2.1. Let (E, d) be a metric space.

Let $p \geq 0$. In the rest of the paper, we will say that $f(\lambda, \pi) \in E$ tends to $\ell \in E$ when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$ if for all $\delta > 0$, there are $\varepsilon > 0$ and $\lambda_0 \in (0, 1]$ such that for all $\lambda \in (0, \lambda_0)$ and all $\pi \geq 1$ in such a way that $\left| \frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p \right| < \varepsilon$, there holds $d(f(\lambda, \pi), \ell) < \delta$.

Let $z_0 \in [0, 1]$. Similarly, we will say that $f(\lambda, \pi) \in E$ tends to $\ell \in E$ when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$ if for all $\delta > 0$, there are $\varepsilon > 0$, $K_0 > 0$ and $\lambda_0 \in (0, 1]$ such that for all $\lambda \in (0, \lambda_0)$ and all $\pi \geq 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \geq K_0$ and $\left| \frac{\log(\pi)}{\log(1/\lambda)} - z_0 \right| < \varepsilon$, there holds $d(f(\lambda, \pi), \ell) < \delta$.

2.3 Main results when $p \in [0, \infty)$

In this section, we are interested in the regime $\mathcal{R}(p)$, for some $p \in [0, \infty)$. We treat together the cases $p = 0$ and $p \in (0, \infty)$. There are just few differences between these two cases: see Remark 2.3 for an alternative definition in the case $p = 0$.

2.3.1 Definition of the limit forest fire process

We now describe the limit process. We want this process to be Markov and this forces us to add some variables. We consider a Poisson measure $\pi_M(dx, dt)$ on $\mathbb{R} \times [0, \infty)$, with intensity measure $dx dt$, whose marks correspond to matches. Recall Notation 2.1.

Definition 2.2. Let $p \geq 0$. A process $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ with values in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}$ such that a.s., for all $x \in \mathbb{R}$, $(Z_t(x), H_t(x))_{t \geq 0}$ is càdlàg, is said to be a p -limit-forest-fire-process (or LFFP(p) in short), if a.s., for all $t \geq 0$, all $x \in \mathbb{R}$,

$$\begin{aligned} Z_t(x) &= \int_0^t \mathbf{1}_{\{Z_s(x) < 1\}} ds - \sum_{s \leq t} (F_s(x) \wedge 1), \\ H_t(x) &= \int_0^t Z_{s-}(x) \mathbf{1}_{\{Z_{s-}(x) < 1\}} \pi_M(\{x\} \times ds) - \int_0^t \mathbf{1}_{\{H_s(x) > 0\}} ds, \\ F_t(x) &= \iint_{(y,s) \in \Lambda_{(x,t)}^p} \mathbf{1}_{\{\forall (r,v) \in \Lambda_{(x,t)}^p(y,s), Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0\}} \pi_M(dy, ds). \end{aligned} \tag{2.8}$$

To the LFFP(p), we associate the process $D_t(x) = [L_t(x), R_t(x)]$, with

$$\begin{aligned} L_t(x) &= \sup\{y \leq x : Z_t(y) < 1 \text{ or } H_t(y) > 0\}, \\ R_t(x) &= \inf\{y \geq x : Z_t(y) < 1 \text{ or } H_t(y) > 0\}. \end{aligned}$$

A typical path of the finite box version of the LFFP(p) is drawn and commented in Figure 3 and a simulation algorithm is explained in the proof of Proposition 3.4.

Remark 2.3. If $p = 0$, we can rewrite the process $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ as follow

$$\begin{aligned} Z_t(x) &= \int_0^t \mathbf{1}_{\{Z_s(x) < 1\}} ds - \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-}(x)=1, y \in D_{s-}(x)\}} \pi_M(dy, ds), \\ H_t(x) &= \int_0^t Z_{s-}(x) \mathbf{1}_{\{Z_{s-}(x) < 1\}} \pi_M(\{x\} \times ds) - \int_0^t \mathbf{1}_{\{H_s(x) > 0\}} ds, \\ F_t(x) &= \int_{\mathbb{R}} \mathbf{1}_{\{Z_{t-}(x)=1, y \in D_{t-}(x)\}} \pi_M(dy \times \{t\}), \end{aligned}$$

where $D_{t-}(x)$ is defined as above. Indeed, for all $x \in \mathbb{R}$, all $t \geq 0$,

$$\left\{ (y, s) : \forall (r, v) \in \Lambda_{(x,t)}^0(y, s) : Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0 \right\} = D_t(x) \times \{t\}$$

With a slightly different formulation, this limit process is the same as in [5] where the propagation is instantaneous. This relationship is very natural. Indeed, the case $p = 0$ corresponds to the case where the propagation velocity is very high.

2.3.2 Formal dynamics

Let us explain the dynamics of this process. For $p \in [0, \infty)$, we consider $T > 0$ fixed and set $\mathcal{A}_T = \{x \in \mathbb{R} : \pi_M(\{x\} \times [0, T]) > 0\}$. For each $t \geq 0$, $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x . We call this cluster *microscopic* if $D_t(x) = \{x\}$. Otherwise, we call it *macroscopic*.

1. *Initial condition.* We have $Z_0(x) = H_0(x) = F_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.
2. *Occupation of vacant zones.* We consider here $x \in \mathbb{R} \setminus \mathcal{A}_T$. Then we have $H_t(x) = 0$ for all $t \in [0, T]$. When $Z_t(x) < 1$, $D_t(x) = \{x\}$ and $Z_t(x)$ stands for the *local density of occupied sites* around x . Then $Z_t(x)$ grows linearly until it reaches 1, as described by the first term on the RHS of the first equation in (2.8). When $Z_t(x) = 1$, the cluster containing x is macroscopic and is described by $D_t(x)$.
3. *Microscopic fires.* Here we assume that $x \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time t where $Z_{t-}(x) < 1$. In such a case, the cluster containing x is microscopic. Then we set $H_t(x) = Z_{t-}(x)$, as described by the first term on the RHS of the second equation of (2.8) and we leave unchanged the value of $Z_t(x)$ and $F_t(x)$. We then let $H_t(x)$ decrease linearly until it reaches 0, see the second term on the RHS of the second equation in (2.8). At all times where $H_t(x) > 0$, that is during $[t, t + Z_{t-}(x))$, the site x acts like a barrier (see Point 4. below).
4. *Macroscopic fires.* Here we assume that $y \in \mathcal{A}_T$ and that the corresponding mark of π_M happens at some time s where $Z_{s-}(y) = 1$. This means that the cluster containing y is macroscopic. Thus this mark creates 2 fires: one goes to the left, the other to the right. These fires propagates along of $\mathcal{V}_{(y,s)}^p$, until they are stopped by a microscopic zone or a barrier or an other fire.
- In other words, for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, we set $F_t(x) = 0$ unless there exists one (or two) mark (y, s) of π_M such that $(y, s) \in \Lambda_{(x,t)}^p$ (or equivalently $(x, t) \in \mathcal{V}_{(y,s)}^p$) and for all $(r, v) \in \Lambda_{(x,t)}^p(y, s)$, $Z_{v-}(r) = 1$ and $H_{v-}(r) = 0$, in which case we set $F_t(x) = 1$ (or $F_t(x) = 2$). When x is crossed by a fire, $Z_t(x)$ jumps from 1 to 0, see the second term on the RHS of the first equation in (2.8).
5. *Clusters.* Finally the definition of the clusters $(D_t(x))_{x \in \mathbb{R}}$ becomes more clear: these clusters are delimited by zones with local density smaller than 1 (i.e. $Z_t(y) < 1$) or by sites where a microscopic fire has (recently) started (i.e. $H_t(y) > 0$).

2.3.3 Well posedness

The existence and uniqueness of the LFFP(0) has been proved in [5]. The proof in the case $p \in (0, \infty)$ is in the same spirit.

Theorem 2.4. *For any Poisson measure $\pi_M(dx, dt)$ on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$, there a.s. exists a unique LFFP(p). Furthermore, it can be constructed graphically and its restriction to any finite box $[0, T] \times [-n, n]$ can be perfectly simulated.*

The LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ is furthermore Markov, since it solves a well-posed time homogeneous Poisson-driven S.D.E.

2.3.4 The convergence result

Theorem 2.5. *Consider for each $\lambda \in (0, 1], \pi \geq 1$, the process $(Z_t^{\lambda, \pi}(x), D_t^{\lambda, \pi}(x))_{t \geq 0, x \in \mathbb{R}}$ associated to the (λ, π) -FFP. Consider also the LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ and the associated $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$. We assume that $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, for some $p \in [0, \infty)$.*

1. *For any $T > 0$, any finite subset $\{x_1, \dots, x_q\} \subset \mathbb{R}$, $(Z_t^{\lambda, \pi}(x_i), D_t^{\lambda, \pi}(x_i))_{t \in [0, T], i=1, \dots, q}$ goes in law to $(Z_t(x_i), D_t(x_i))_{t \in [0, T], i=1, \dots, q}$ in $\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))$. Here $\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))$ is endowed with the distance \mathbf{d}_T .*
2. *For any finite subset $\{(x_1, t_1), \dots, (x_q, t_q)\} \subset \mathbb{R} \times [0, \infty)$, $(Z_{t_i}^{\lambda, \pi}(x_i), D_{t_i}^{\lambda, \pi}(x_i))_{i=1, \dots, q}$ goes in law to $(Z_{t_i}(x_i), D_{t_i}(x_i))_{i=1, \dots, q}$ in $(\mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))^q$. Here $\mathcal{I} \cup \{\emptyset\}$ is endowed with δ .*
3. *For all $t > 0$,*

$$\left(\frac{\log(|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)| \geq 1\}} \right) \wedge 1$$

goes in law to $Z_t(0)$.

Point 3 will allow us to check some estimates on the cluster-size distribution. Since we deal with finite-dimensional marginals in space, it is quite clear that the processes H and F do not appear in the limit, since for each $x \in \mathbb{R}$, for all $t \geq 0$, a.s., $H_t(x) = F_t(x) = 0$. (of course, it is false that a.s., for all $x \in \mathbb{R}$, all $t \geq 0$, $H_t(x) = F_t(x) = 0$). We obtain the convergence of $D^{\lambda, \pi}$ (resp. $Z^{\lambda, \pi}$) to D (resp. Z) only when integrating in time. We cannot hope for a Skorokhod convergence since the limit process $D(x)$ (resp. $Z(x)$) jumps instantaneously from $\{x\}$ (resp. 1) to some interval with positive length (resp. 0), while $D^{\lambda, \pi}(x)$ (resp. $Z^{\lambda, \pi}(x)$) needs many small jumps, in a very short interval, to become macroscopic (resp. empty).

The space $(\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\})), \mathbf{d}_T)$ is not a complete metric space since \mathbf{d}_T is too weak. However, it seems that it is not really a problem because in the proof, we use a coupling argument and obtain a convergence in probability.

2.3.5 Heuristics argument

We now explain roughly the reasons why Theorem 2.5 holds. We consider a (λ, π) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and the associated process $(Z_t^{\lambda, \pi}(x), D_t^{\lambda, \pi}(x))_{t \geq 0, x \in \mathbb{R}}$. We assume below that λ is very small, π very large and $\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi)$ close to p .

0. *Scales.* With our scales, there are $\mathbf{n}_\lambda = \lfloor 1/(\lambda \log(1/\lambda)) \rfloor$ sites per unit of length. Approximately one fire starts per unit of time per unit of length. A vacant site becomes occupied at rate $\mathbf{a}_\lambda = \log(1/\lambda)$.

1. *Initial condition.* We have, for all $x \in \mathbb{R}$, $(Z_0^{\lambda, \pi}(x), D_0^{\lambda, \pi}(x)) = (0, \emptyset) \simeq (0, \{x\})$.

2. *Occupation of vacant zones.* Assume that no match falls in a zone $[a, b]$ (which correspond to the zone $[\lfloor \mathbf{n}_\lambda a \rfloor, \lfloor \mathbf{n}_\lambda b \rfloor]$ before rescaling) during $[0, 1]$ (or $[0, \mathbf{a}_\lambda]$ before rescaling).

a. For $s \in [0, 1]$, we have $D_s^{\lambda, \pi}(x) \simeq [x \pm \lambda^{1-s}] \simeq \{x\}$ and $Z_s^{\lambda, \pi}(x) \simeq s$ for all $x \in [a, b]$.

Indeed, each site is occupied with probability $1 - e^{-\mathbf{a}_\lambda s} = 1 - \lambda^s$. Thus the local density is roughly $K_t^{\lambda, \pi} \simeq 1 - \lambda^s$, whence $Z_t^{\lambda, \pi}(x) \simeq s$, while the typical size of occupied clusters is λ^s , whence $D_s^{\lambda, \pi}(x) \simeq [x \pm \lambda^s / \mathbf{n}_\lambda] \simeq [x \pm \lambda^{1-s}]$.

- b. At time $s = 1$, $Z_1^{\lambda, \pi}(x) \simeq 1$ and all the sites in $[a, b]$ are occupied (with very high probability).

Indeed, we have $(b - a)\mathbf{n}_\lambda$ sites and each of them is occupied at time 1 with probability $1 - e^{-\mathbf{a}_\lambda} = 1 - \lambda$ so that all of them are occupied with probability $(1 - \lambda)^{(b-a)\mathbf{n}_\lambda} \simeq e^{-(b-a)/\log(1/\lambda)}$, which goes to 1 as $\lambda \rightarrow 0$.

Assume now that the zone around x (i.e. the zone $[[\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]]$ before rescaling) has been destroyed at time t (or at time $\mathbf{a}_\lambda t$ before rescaling) by a fire. Then, observations 2a. and 2b. above still hold:

- i. for $s \in [0, 1)$ and if no fire starts in $[[\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda]]$ during $[\mathbf{a}_\lambda t, \mathbf{a}_\lambda(t + s)]$, we have $D_{t+s}^{\lambda, \pi}(x) \simeq [x \pm \lambda^{1-s}] \simeq \{x\}$ and $Z_{t+s}^{\lambda, \pi}(x) \simeq s$;

- ii. $Z_{t+1}^{\lambda, \pi}(x) \simeq 1$ and all the sites around x are occupied at time $t + 1$ with very high probability.

3. *Microscopic fires.* Assume that a fire starts at some location x (i.e. $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at some time t (or $\mathbf{a}_\lambda t$ before rescaling) with $Z_{t-}^{\lambda, \pi}(x) = z \in (0, 1)$. The possible clusters on the left and right of x cannot be connected during (approximately) $[t, t + z]$, but they can be connected after (approximately) $t + z$. In other words, x acts like a barrier during $[t, t + z]$.

Indeed, the connected component A of x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at time t (or $\mathbf{a}_\lambda t$ before rescaling) has a size of order λ^{1-z} (which thus contains approximately $\lambda^{1-z}\mathbf{n}_\lambda \simeq \lambda^{-z}$ sites). The fire destroys the component A in a time of order $1/(\lambda^z \mathbf{a}_\lambda \pi) \ll 1$ (or $1/(\lambda^z \pi) \ll \mathbf{a}_\lambda$ in original scale). Thus this fire crosses very fast the component A and each site of A becomes burning and then empty (i.e. $\eta^{\lambda, \pi}(i)$ jumps from 1 to 2 then from 2 to 0) during the time interval $[t, t + 1/(\lambda^z \mathbf{a}_\lambda \pi)] \simeq \{t\}$ (or $[\mathbf{a}_\lambda t, \mathbf{a}_\lambda t + 1/(\lambda^z \pi)] \simeq \{\mathbf{a}_\lambda t\}$ before rescaling). The probability that a fire starts again in A is very small. Thus, using the same computation as in point 2, we observe that $\mathbb{P}[A \text{ is completely occupied at time } t + s] \simeq (1 - \lambda^s)^{\lambda^{-z}} \simeq e^{-\lambda^{s-z}}$. When $\lambda \rightarrow 0$, this quantity tends to 0 if $s < z$ and to 1 if $s > z$.

4. *Macroscopic fires.* Assume, now, that a fire starts at some place x (i.e. $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at some time t (or $\mathbf{a}_\lambda t$ before rescaling) and that $Z_{t-}^{\lambda, \pi}(x) \simeq 1$. Thus, $D_{t-}^{\lambda, \pi}(x)$ is macroscopic (i.e. its length is of order 1 in our scales). Then the match creates two fires: one propagates to the left and one to the right at speed p (p unit times per unit space). There are only two burning trees at each instant with very high probability. Of course, these fires are stopped when they meet a vacant site (i.e. a microscopic zone or a barrier) or another fire.

Indeed, we have to wait for an exponential time of parameter π between each propagation in the original scales. It then produces two independent Poisson processes of parameter π which stand for the location of the fires. Then, for $b > x$, this Poisson process is at $\lfloor \mathbf{n}_\lambda b \rfloor$ in the original scale (or in b after rescaling) roughly at time $\mathbf{a}_\lambda t + (\mathbf{n}_\lambda / \pi)(b - x)$ (or at time $t + (\mathbf{n}_\lambda / (\mathbf{a}_\lambda \pi))(b - x) \simeq t + p(b - x)$ after rescaling). All sites $i \in [[\lfloor \mathbf{n}_\lambda x \rfloor, \lfloor \mathbf{n}_\lambda b \rfloor]]$ becomes successively burning and empty roughly at time $\mathbf{a}_\lambda t + (i - \lfloor \mathbf{n}_\lambda x \rfloor) / \pi$ in the original scale (or the site $y = i / \mathbf{n}_\lambda \in \mathbb{R}$ is burning at time $t + p(y - x)$ after rescaling).

5. *Clusters.* For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D_t^{\lambda, \pi}(x)$ resembles $[x \pm \lambda^{1-z}] \simeq \{x\}$ if $Z_t^{\lambda, \pi}(x) = z \in (0, 1)$. We then say that x is microscopic. Now, macroscopic clusters are delimited either by microscopic zones or by sites where there has been recently a microscopic fire (see point 3) or by a burning tree.

Comparing the arguments above to the rough description of the LFFP(p) (see Section 2.3.2), our hope is that the (λ, π) -FFP resembles the LFFP(p) for $\lambda > 0$ very small, π very large and $1/(\lambda \mathbf{a}_\lambda^2 \pi)$ close to p .

Remark 2.6. Remark 2.3 is now more clear. Consider the regime $\mathcal{R}(0)$. If a fire starts at x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at time t (or $\mathbf{a}_\lambda t$ before rescaling), the time needed to reach a point b (or $\lfloor \mathbf{n}_\lambda b \rfloor$ before rescaling) is roughly $\mathbf{n}_\lambda |b - x| / (\mathbf{a}_\lambda \pi) \simeq 0$ (or $\mathbf{n}_\lambda (b - x) / \pi \ll \mathbf{a}_\lambda$ before rescaling). It means that if $b \in D_{t-}^0(x)$ (or $\lfloor \mathbf{n}_\lambda b \rfloor \in C(\eta_{\mathbf{a}_\lambda t-}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x \rfloor)$ before rescaling) the fire reaches b at time $t + \mathbf{n}_\lambda |b - x| / (\mathbf{a}_\lambda \pi) \simeq t$. In the scaling limit, the cluster containing x is thus destroyed instantaneously.

2.3.6 Cluster size distribution

We will deduce from Theorem 2.5 the following estimates on the cluster-size distribution.

Corollary 2.7. *Let $p \in [0, \infty)$ be fixed. Let $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(p) and $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ the associated process. For each $\lambda \in (0, 1]$ and $\pi \geq 1$, let $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ be a (λ, π) -FFP.*

- a. *For all $t \geq (5+p)/2$, all $0 < a < b < 1$, for some $0 < c_1 < c_2$ depending on p , as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$,*

$$\lim_{\lambda, \pi} \mathbb{P} \left[\left| C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0) \right| \in [1/\lambda^a, 1/\lambda^b] \right] = \mathbb{P} [Z_t(0) \in [a, b]] \in [c_1(b-a), c_2(b-a)].$$

- b. *For all $t \geq 3/2$, all $B > 0$, for some $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ depending on p , as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$,*

$$\lim_{\lambda, \pi} \mathbb{P} \left[\left| C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0) \right| \geq B n_\lambda \right] = \mathbb{P} [|D_t(0)| \geq B] \in [c_1 e^{-\kappa_2 B}, c_2 e^{-\kappa_1 B}].$$

This result shows that there is a *phase transition* around the critical size \mathbf{n}_λ : the cluster-size distribution changes of shape at \mathbf{n}_λ . The main idea is that two types of clusters are present: macroscopic clusters, of which the size is of order \mathbf{n}_λ and microscopic clusters, of which the size is smaller than \mathbf{n}_λ .

2.4 Main results for $p = \infty$

In this section, we are interested in the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in [0, 1]$.

2.4.1 Definition of the limit process

In this regime, the limit process is much simpler, in the sense that fires only have a local (in space) effect (but can have long time effect). This is due to the fact that a fire can't go too far away in a finite time.

We consider a Poisson measure $\pi_M(dx, dt)$ on $\mathbb{R} \times [0, \infty)$, with intensity measure $dx dt$, whose marks correspond to matches.

Definition 2.8. *Let $z_0 \in [0, 1]$. A process $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$ with values in \mathbb{R}_+ such that a.s., for all $x \in \mathbb{R}$, $(Y_t(x))_{t \geq 0}$ is càdlàg, is said to be a LFFP(∞, z_0) if a.s., for all $t \geq 0$, all $x \in \mathbb{R}$,*

$$Y_t(x) = \int_0^{t \wedge z_0} s \pi_M(\{x\} \times ds) - \int_0^t \mathbf{1}_{\{Y_s(x) \in [0, 1]\}} ds + \mathbf{1}_{\{t \geq z_0\}} \pi_M(\{x\} \times [z_0, t]). \quad (2.9)$$

The process Y takes its values in $[0, 1]$ and can be non-zero only at locations where $\pi_M(\{x\} \times \mathbb{R}) \neq 0$. If the mark of π_M happens at time $t < z_0$, then the (microscopic) cluster containing x is destroyed instantaneously and $Y_s(x) \in (0, 1)$ during $[t, 2t]$: x acts like a barrier during this time interval. If the mark happens at time $t > z_0$ then the cluster containing x is too big to be destroyed and $Y_s(x) = 1$ for ever: there is always a burning tree close to x . We then naturally associate the process $D_t(x) = [L_t(x), R_t(x)]$, with

$$L_t(x) = \begin{cases} x & \text{if } t < 1, \\ \sup\{y \leq x : Y_t(y) > 0\} & \text{if } t \geq 1; \end{cases}$$

$$R_t(x) = \begin{cases} x & \text{if } t < 1, \\ \inf\{y \geq x : Y_t(y) > 0\} & \text{if } t \geq 1. \end{cases}$$

A typical path of the finite box version of the LFFP(∞, z_0) is drawn and commented in Figure 2.

Remark 2.9. *The process Y is a time inhomogeneous Markov process. To make it homogeneous, we can add a second variable Z as in the first equation (2.8) in the Definition 2.2.*

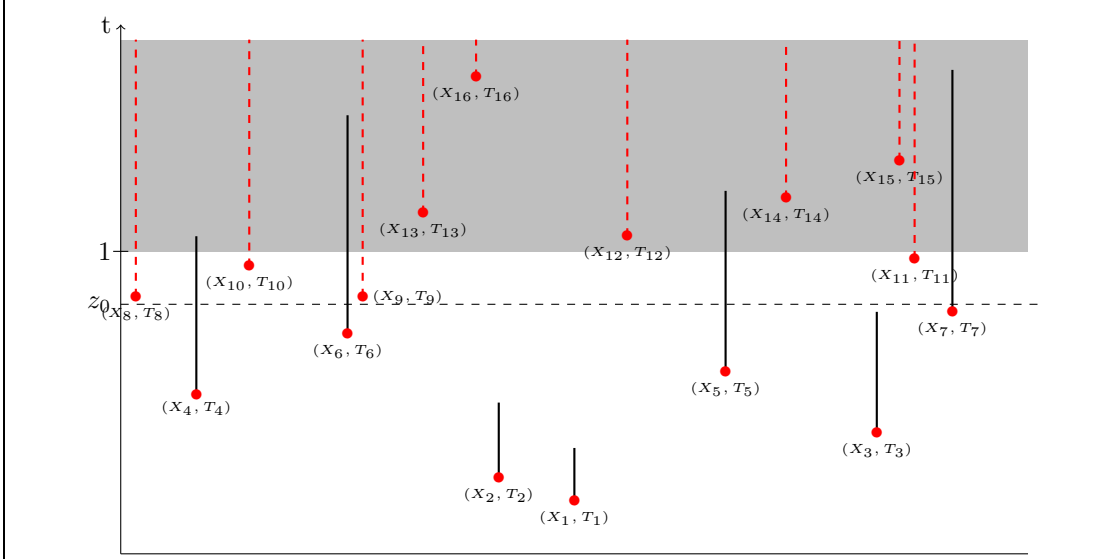


Figure 2: LFF(∞, z_0)–process in a finite box.

The marks of π_M are represented by \bullet 's. The filled zones represents zones in which $|D(x)| > 0$. The plain vertical segments represent the sites where $Y_t(x) \in (0, 1)$ and the dashed vertical segments represent the sites where $Y_t(x) = 1$. In the rest of the space, we always have $Y_t(x) = 0$. Until time 1, all the particles are microscopic. Matches 1 to 7 falls before z_0 . At each of these marks, a process Y starts and its life-time equals the instant where it has started. This creates a barrier with height T_k (the segment above T_k ends at time $2T_k$). The other matches falls after z_0 . At each of these marks, a process Y starts and remains equal to 1 forever.

Thus, for each $x \in [-A, A]$, $D_t^A(x) = \{x\}$ for $t \in [0, 1)$ and merge at $t = 1$. Here we have at time 1 the clusters $[-A, X_8]$, $[X_8, X_4]$, $[X_4, X_{10}]$, $[X_{10}, X_6]$, $[X_6, X_9]$, $[X_9, X_5]$, $[X_5, X_{11}]$, $[X_{11}, X_7]$ and $[X_7, A]$.

Remark that $t \mapsto |D_t(x)|$ is non-increasing on $[2z_0, \infty)$ for all x .

2.4.2 Formal dynamics

Let us explain the dynamics of this process. We consider $\mathcal{A} = \{x \in \mathbb{R} : \pi_M(\{x\} \times [0, \infty)) > 0\}$. For each $t \geq 0$, $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x . We call this cluster *microscopic* if $D_t(x) = \{x\}$. Otherwise, we call it *macroscopic*.

1. *Initial condition.* We have $Y_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.
2. *Occupation of vacant zones.* We consider here $x \in \mathbb{R} \setminus \mathcal{A}$. Then we have $Y_t(x) = 0$ for all $t \in [0, \infty)$. When $t < 1$, $D_t(x) = \{x\}$. When $t \geq 1$, the cluster containing x is macroscopic and is described by $D_t(x)$.
3. *First kind of fires.* Here we assume that $x \in \mathcal{A}$ and that the corresponding mark of π_M happens at some time $t < z_0$. We set $Y_t(x) = t$, as described by the first term on the RHS of the equation of (2.9). We then let $Y_t(x)$ decrease linearly until it reaches 0, see the second term on the RHS of the equation in (2.9) (i.e. $Y_s(x) = \min(2t - s, 0)\mathbf{1}_{\{s \geq t\}}$).
4. *Second kind of fires.* Here we assume that $x \in \mathcal{A}$ and that the corresponding mark of π_M happens at some time t where $t > z_0$. Then we set $Y_s(x) = 1$ for all $s \in [t, \infty)$ see the third term of the RHS of the equation (2.9).
5. *Clusters.* Finally the definition of the clusters $(D_t(x))_{x \in \mathbb{R}}$ becomes more clear: these clusters remain microscopic until $t = 1$. For $t \geq 1$, $(D_t(x))_{x \in \mathbb{R}, t \geq 1}$ is delimited by sites where a fire of first kind has (recently) started (i.e. $Y_t(y) \in (0, 1)$) or by sites where a fire of second kind has started (i.e. $Y_t(y) = 1$). Remark that for $t \geq 2z_0$, only fires of second kind delimit the clusters.

2.4.3 Well posedness

The following proposition is obvious from the definition, see Figure 2.

Proposition 2.10. *Let π_M be a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$.*

There a.s. exists a unique LFFP(∞, z_0) $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$. It can be simulated exactly on any finite box $[0, T] \times [-n, n]$.

2.4.4 The convergence result

We will prove the following result.

Theorem 2.11. *Let $z_0 \in [0, 1]$. Consider for each $\lambda \in (0, 1]$ and $\pi \geq 1$ the process $(D_t^{\lambda, \pi}(x))_{t \geq 0, x \in \mathbb{R}}$ associated with the (λ, π) -FFP. Consider also the LFFP(∞, z_0) $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$ and the associated $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ process. We assume that $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the slow regime $\mathcal{R}(\infty, z_0)$.*

1. *For any $T > 0$, any finite subset $\{x_1, \dots, x_q\} \subset \mathbb{R}$, $(D_t^{\lambda, \pi}(x_i))_{t \in [0, T], i=1, \dots, q}$ goes in law to $(D_t(x_i))_{t \in [0, T], i=1, \dots, q}$ in $\mathbb{D}([0, T], \mathcal{I})^q$. Here $\mathbb{D}([0, T], \mathcal{I})^q$ is endowed with δ_T .*
2. *For any finite subset $\{(x_1, t_1), \dots, (x_q, t_q)\} \subset \mathbb{R} \times [0, \infty)$, $(D_{t_i}^{\lambda, \pi}(x_i))_{i=1, \dots, q}$ goes in law to $(D_{t_i}(x_i))_{i=1, \dots, q}$ in \mathcal{I}^q , \mathcal{I} being endowed with δ .*

2.4.5 Heuristics arguments

We assume below that $\lambda > 0$ is very small, $\pi \geq 1$ is very large, $\lambda \mathbf{a}_\lambda^2 \pi$ is close to 0 and $\log(\pi)/\log(1/\lambda)$ is close to z_0 .

0. *Scales.* With our scales, there are $\mathbf{n}_\lambda = \lfloor 1/(\lambda \log(1/\lambda)) \rfloor$ sites per unit of length. Approximately one fire starts per unit of time per unit of length. A vacant site becomes occupied at rate $\mathbf{a}_\lambda = \log(1/\lambda)$.

1. *Initial condition.* We have, for all $x \in \mathbb{R}$, $D_0^{\lambda, \pi}(x) = \emptyset \simeq \{x\}$ and $D_0(x) = \{x\}$.
2. *Occupation of vacant zones.* Exactly as in the regime $\mathcal{R}(p)$, $D_t^{\lambda, \pi}(x) \simeq [x \pm \lambda^{1-t}] \simeq \{x\}$ for $t < 1$ and the clusters become macroscopic at time 1.
3. *First kind of fires.* Assume that a match falls at some place x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ in the original scales) at some time $t < z_0$ (or $\mathbf{a}_\lambda t < \mathbf{a}_\lambda z_0$ in the original scales). Then the fire burns almost immediately the occupied cluster and it needs roughly a time t (or $\mathbf{a}_\lambda t$ in the original scales) to be filled again. Thus x acts like a barrier during $[t, 2t)$.

Indeed, the connected component A of x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ before rescaling) at time t (or $\mathbf{a}_\lambda t$ before rescaling) has a size of order λ^{1-t} (which thus contains approximately $\lambda^{1-t} \mathbf{n}_\lambda \simeq \lambda^{-t}$ sites). The fire destroys the component A in a time of order $1/(\lambda^t \mathbf{a}_\lambda \pi) \ll 1$ (or $1/(\lambda^t \pi) \ll \mathbf{a}_\lambda$ in original scales) due to $\mathcal{R}(\infty, z_0)$. Thus this fire crosses very fast the component A and each site of A becomes burning and then empty (i.e. $\eta^{\lambda, \pi}(i)$ jumps from 1 to 2 then from 2 to 0) during the time interval $[t, t + 1/(\lambda^t \mathbf{a}_\lambda \pi)] \simeq \{t\}$ (or $[\mathbf{a}_\lambda t, \mathbf{a}_\lambda t + 1/(\lambda^t \pi)] \simeq \{\mathbf{a}_\lambda t\}$ before rescaling). The probability that a fire starts again in A is very small. Thus, we observe that $\mathbb{P}[A \text{ is completely occupied at time } t + s] \simeq (1 - \lambda^s)^{\lambda^{-z}} \simeq e^{-\lambda^{s-z}}$. When $\lambda \rightarrow 0$, this quantity tends to 0 if $s < t$ and to 1 if $s > t$.

4. *Second kind of fires.* Assume that a match falls at some place x (or $\lfloor \mathbf{n}_\lambda x \rfloor$ in the original scales) at some time $t > z_0$ (or $\mathbf{a}_\lambda t > \mathbf{a}_\lambda z_0$ in the original scales). Then the fire needs an infinite time (in our scales) to burn the occupied cluster, so that there is a burning site close to x forever.

Indeed, $D_t^{\lambda, \pi}(x)$ contains roughly λ^{-t} sites if $t \in (z_0, 1)$ and \mathbf{n}_λ sites if $t \geq 1$. In any case, the time needed for the fire to cross this cluster is of order $|D_t^{\lambda, \pi}(x)|/\pi$, which is very large when compared to \mathbf{a}_λ in the regime $\mathcal{R}(\infty, z_0)$. Thus, the fire cannot reach the rim of $D_t^{\lambda, \pi}(x)$.

5. *Clusters.* For $t \geq 0$, $x \in \mathbb{R}$, the cluster $D_t^{\lambda, \pi}(x)$ resembles $[x \pm \lambda^{1-z}] \simeq \{x\}$ if $t < 1$. Now, macroscopic clusters emerge when $t \geq 1$ and are delimited either by a burning tree or by sites where there has been recently a microscopic fire (see point 3).

Comparing the arguments above to the rough description of the LFFP(∞, z_0) (see Section 2.4.2), our hope is that the (λ, π) -FFP resembles the LFFP(∞, z_0) in the regime $\mathcal{R}(\infty, z_0)$.

2.4.6 Cluster-size distribution

The following corollary is easily deduced from the Theorem 2.11.

Corollary 2.12. *Let $z_0 \in [0, 1]$. Let $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(∞, z_0) and $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ the associated process. For each $\lambda \in (0, 1]$ and $\pi \geq 1$, let $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ be a (λ, π) -FFP.*

For all $t > 2z_0$, as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$,

$$\frac{1}{\mathbf{n}_\lambda} \left| C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0) \right| \xrightarrow{\mathcal{L}} |D_t(0)| \sim \Gamma(2, t - z_0).$$

This result shows that for t large enough, there are only macroscopic clusters, that is clusters with size of order \mathbf{n}_λ .

We immediately give the proof of Corollary 2.12. For $t \geq 0$, Theorem 2.11 shows that, when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$,

$$\frac{1}{\mathbf{n}_\lambda} \left| C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0) \right| \xrightarrow{\mathcal{L}} |D_t(0)|.$$

Furthermore, if $t > 2z_0$, only fires of the second kind (i.e. matches falling after z_0) still have an effect. Indeed, when a match falls in x at time $t < z_0$, it creates a barrier in x during $[t, 2t) \subset [0, 2z_0]$. Thus, $D_t(0)$ is only delimited by sites where a match has fallen during $[z_0, t]$. This is a Poisson process on \mathbb{R} with intensity $t - z_0$. Consequently,

$$|D_t(0)| \sim \Gamma(2, t - z_0).$$

2.4.7 Irreversibility

It might look surprising at the first glance that the limit process is non-reversible while the discrete process is reversible. Indeed, for $t \geq 1 \wedge 2z_0$, clusters in the limit process are macroscopic and the sizes are non-increasing. On the other hand, in the discrete process, it is quite clear that, when working in a finite box, the process returns to its original state. This is due to the time scale: we have to wait a very long time to observe again the original state.

3 Existence and uniqueness of the limit process

The goal of this section is to show that the limit processes are well-defined, unique, can be obtained from a graphical construction and can be restricted to a finite box.

3.1 Restriction of the LFFP(∞, z_0) to a finite box

Let $z_0 \in [0, 1]$ be fixed. In this subsection, we study the LFFP(∞, z_0).

Proposition 3.1. *Let π_M a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$ and $A > 0$.*

1. *The values of $(Y_t(x))_{t \geq 0, x \in [-A, A]}$ are entirely determined by $\pi_M|_{[-A, A] \times \mathbb{R}_+}$. Actually, for all $x \in \mathbb{R}$, the values of $(\tilde{Y}_t(x))_{t \geq 0}$ are entirely determined by $\pi_M|_{\{x\} \times \mathbb{R}_+}$.*
2. *There exists some constants $\alpha > 0$ and $C > 0$ not depending on $A > 0$ such that*

$$\mathbb{P} \left[(D_t(x))_{t \geq 0, x \in [-A/2, A/2]} \subset [-A, A] \right] \geq 1 - Ce^{-\alpha A}. \quad (3.1)$$

Proof. The first part of Proposition 3.1 is obvious from the definition of the process $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$. In order to prove the second part, consider the event Ω_A^+ on which π_M has at least one mark (X_1, τ_1) in $[A/2, A] \times (3/4, 1)$ and at least one mark (X_2, τ_2) in $[A/2, A] \times (1, 3/2)$.

Observe now that on Ω_A^+ , $Y_t(X_1) > 0$ for all $t \in [\tau_1, 2\tau_1) \supset [1, 3/2]$, because it is either a fire of first kind (if $\tau_1 \leq z_0$) or X_1 burns for ever (if $\tau_1 > z_0$), and $Y_t(X_2) = 1$ for all $t \in [\tau_2, \infty) \supset [3/2, \infty)$ because $\tau_2 > 1 \geq z_0$, then X_2 burns for ever, because it is necessarily a fire of second kind.

Similarly, we define the event Ω_A^- on which π_M has at least one mark $(\tilde{X}_1, \tilde{\tau}_1)$ in $[-A, -A/2] \times (3/4, 1)$ and at least one mark $(\tilde{X}_2, \tilde{\tau}_2)$ in $[-A, -A/2] \times (1, 3/2)$.

Thus, on $\Omega_A^+ \cap \Omega_A^-$, $D_t(x) \subset [-A, A]$ for all $t \geq 0$ and all $x \in [-A/2, A/2]$. Finally, we can bound from below the left hand side of (3.1) by

$$\mathbb{P}[\Omega_A^+ \cap \Omega_A^-] \geq 1 - 2(e^{-A/8} + e^{-A/4}) \geq 1 - 4e^{-A/8}$$

whence (3.1) with $C = 4$ and $\alpha = 1/8$. \square

Definition 3.2. Let $z_0 \in [0, 1]$ and $(Y_t(x))_{x \in \mathbb{R}, t \geq 0}$ be a LFFP(∞, z_0). For all $A > 0$ and for $x \in [-A, A]$, we define the process $D_t^A(x) = [L_t^A(x), R_t^A(x)]$, with

$$\begin{aligned} L_t^A(x) &= \begin{cases} x & \text{if } t < 1, \\ \sup\{y \leq x : Y_t(y) > 0\} \vee (-A) & \text{if } t \geq 1; \end{cases} \\ R_t^A(x) &= \begin{cases} x & \text{if } t < 1, \\ \inf\{y \geq x : Y_t(y) > 0\} \wedge A & \text{if } t \geq 1. \end{cases} \end{aligned}$$

As a corollary of Proposition 3.1, we have, for $A > 0$,

$$\mathbb{P}[(D_t(x))_{t \geq 0, x \in [-A/2, A/2]} = (D_t^A(x))_{t \geq 0, x \in [-A/2, A/2]}] \geq 1 - Ce^{-\alpha A}.$$

3.2 Restriction of the LFFP(p) to a finite box

The aim of this subsection is to prove Theorem 2.4. We define an analogous process of LFFP(p) on a finite space interval, which can be perfectly simulated. We then show that these two processes are equal with very high probability.

3.2.1 Algorithm

Let $p \in [0, \infty)$. Here we show that when working on a finite space interval, the LFFP(p) is somewhat discrete. We consider a Poisson measure $\pi_M(dx, dt)$ on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$.

Definition 3.3. Let $A > 0$. A process $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t \geq 0, x \in [-A, A]}$ with values in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}$ such that a.s., for all $x \in [-A, A]$, $(Z_t^A(x), H_t^A(x))_{t \geq 0}$ is càdlàg, is a A -LFFP(p) if a.s., for all $t \geq 0$, all $x \in [-A, A]$,

$$\begin{aligned} Z_t^A(x) &= \int_0^t \mathbf{1}_{\{Z_s^A(x) < 1\}} ds - \sum_{s \leq t} (F_s^A \wedge 1), \\ H_t^A(x) &= \int_0^t Z_{s-}^A(x) \mathbf{1}_{\{Z_{s-}^A(x) < 1\}} \pi_M(\{x\} \times ds) - \int_0^t \mathbf{1}_{\{H_s^A(x) > 0\}} ds, \\ F_t^A(x) &= \iint_{(y, s) \in \Lambda_{(x, t)}^p \cap ([-A, A] \times [0, \infty))} \mathbf{1}_{\{\forall (r, v) \in \Lambda_{(x, t)}^p(y, s), Z_{v-}^A(r) = 1 \text{ and } H_{v-}^A(r) = 0\}} \pi_M(dy, ds). \end{aligned} \tag{3.2}$$

To the A -LFFP(p), as usual, we associate the process $D_t^A(x) = [L_t^A(x), R_t^A(x)]$, with

$$\begin{aligned} L_t^A(x) &= (-A) \vee \sup\{y \in [-A, x] : Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}, \\ R_t^A(x) &= A \wedge \inf\{y \in [x, A] : Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}. \end{aligned}$$

A typical path of $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t \geq 0, x \in [-A, A]}$ is drawn in figure 3.

The proof of the following proposition shows the construction of the A -LFFP(p) in an algorithmic way.

Proposition 3.4. Consider a Poisson measure $\pi_M(dx, dt)$ on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$. For any $A > 0$ and $p \geq 0$, there a.s. exists a unique A -LFFP(p) which can be perfectly simulated.

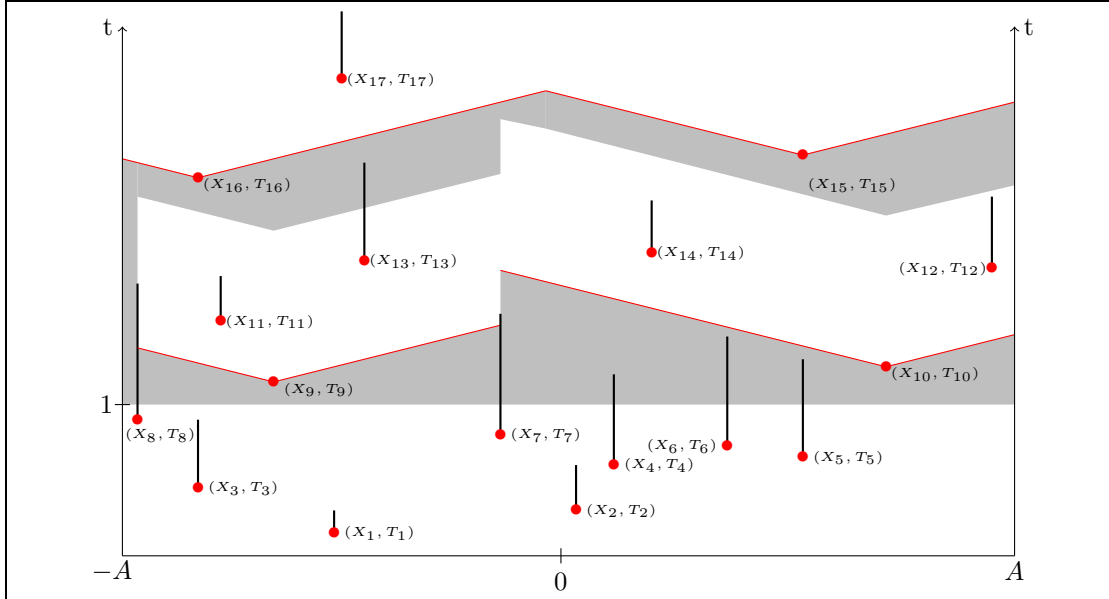


Figure 3: LFFP(p) in a finite box

The marks of π_M (matches) are represented as \bullet 's. The filled zones represent zones in which $Z_t^A(x) = 1$, that is macroscopic clusters. In the rest of the space, we always have $Z_t^A(x) < 1$. The plain vertical segments represent the sites where $H_t^A(x) > 0$. $F_t^A(x) = 0$ except on the lines with slope p where $F_t^A(x) = 1$ or $F_t^A(x) = 2$ in the crossing point of the fires starting in (X_{15}, T_{15}) and (X_{16}, T_{16}) . Until time 1, all of the clusters are microscopic. The first eighth marks of the Poisson measure fall in that zone. As a consequence, at each of these marks, a process H^A starts. Their lifetime is equal to the instant where they have started (e.g., the segment above (X_1, T_1) ends at time $2T_1$). At time 1, all clusters where there has been no mark become macroscopic and merge together. However, this is limited by vertical segments. Here, at time 1, we have the clusters $[-A, X_8]$, $[X_8, X_7]$, $[X_7, X_4]$, $[X_4, X_6]$, $[X_6, X_5]$ and $[X_5, A]$. The segment above (X_4, T_4) ends at time $2T_4$ and thus, at this time, the clusters $[X_7, X_4]$ and $[X_4, X_6]$ merge into $[X_7, X_6]$. The ninth mark falls in the (macroscopic) zone $[X_8, X_7]$ and thus two fires start. They cross the cluster $[X_8, X_7]$ at speed p , i.e. cross $[X_8, X_7]$ with a slope p . A process H^A then starts at X_{11} at time T_{11} . Since $Z_{T_{11}-}^A(X_{11}) = T_{11} - (T_9 + p|X_9 - X_{11}|)$ [because $Z_{T_9+p|X_9-X_{11}|}^A(X_{11})$ has been set to 0], the segment above (X_{11}, T_{11}) will end at time $2T_{11} - (T_9 + p|X_9 - X_{11}|)$. On the other hand, a fire starts at X_{10} at time T_{10} and crosses the cluster of X_{10} at speed p . A site x in $[X_7, A]$ remains microscopic from time $T_{10} + p|X_{10} - x|$ until time $T_{10} + p|X_{10} - x| + 1$. The two matches 14 and 12 create microscopic fires (because they fall on sites where $Z_t^A(x) < 1$). Observe finally that the 15th and the 16th fires are stopped by each other. With this realization, we have $0 \in (X_7, X_2)$ and, thus, $Z_t^A(0) = t$ for $t \in [0, 1]$, then $Z_t^A(0) = 1$ for $t \in [1, T_{10} + pX_{10}]$, then $Z_t^A(0) = t - (T_{10} + pX_{10})$ for $t \in [T_{10} + pX_{10}, T_{10} + pX_{10} + 1]$, then $Z_t^A(0) = 1$ for $t \in [T_{10} + pX_{10} + 1, T_{16} + pX_{15}]$, etc. We also see that $D_t^A(0) = \{0\}$ for $t \in [0, 1]$, $D_t^A(0) = [X_7, X_4]$ for $t \in [1, 2T_4]$, $D_t^A(0) = [X_7, X_6]$ for $t \in [2T_4, 2T_6]$, $D_t^A(0) = [X_7, X_{10} + \frac{T_{10}-t}{p}]$ for $t \in [2T_6, T_{10} + pX_{10}]$, $D_t^A(0) = \{0\}$ for $t \in [T_{10} + pX_{10}, T_{10} + pX_{10} + 1]$, etc. We finally have $F_t^A(0) = 0$ for all $t \neq \{T_{10} + pX_{10}, T_{15} + pX_{15}\}$ and $F_{T_{10}+pX_{10}}^A(0) = F_{T_{15}+pX_{15}}^A(0) = 1$.

Algorithm. Here we only treat the case $p > 0$. The case $p = 0$ is much easier and has been treated in [5], as mentioned in Remark 2.3.

Consider the marks $(X_k, T_k)_{k=1, \dots, n}$ of π_M in $[-A, A] \times [0, T]$, ordered chronologically and set $T_0 = 0$. We describe the construction via an algorithm, which also shows uniqueness, in the sense that there is no choice in the construction.

Suppose that we have built the process $(Z_t^A(x), H_t^A(x), F_t^A(x))_{x \in [-A, A]}$ at some time $t \geq 0$. We

then can set

$$\begin{aligned}\chi_t^+ &= \{x \in [-A, A] : F_t^A(x) = 1 \text{ and } Z_t^A(x+) = 1\}, \\ \chi_t^- &= \{x \in [-A, A] : F_t^A(x) = 1 \text{ and } Z_t^A(x-) = 1\}, \\ \chi_t^0 &= \{x \in [-A, A] : H_t^A(x) > 0 \text{ or } Z_t^A(x+) \neq Z_t^A(x-)\} \cup \{-A, A\}, \\ \chi_t &= \chi_t^+ \cup \chi_t^- \cup \chi_t^0,\end{aligned}$$

where $Z_t^A(x+) = \lim_{y \rightarrow x, y > x} Z_t^A(y)$ (resp. $Z_t^A(x-) = \lim_{y \rightarrow x, y < x} Z_t^A(y)$). Observe that χ_t^+ (resp. χ_t^-) is the set of fires at time t that spread to the right (resp. to the left) and that χ_t^0 is the set of sites where a fire can be stopped (barrier or microscopic zone). We also define, for $r > t$,

$$\mathcal{E}_t^r := \bigcup_{x \in \chi_t^+, y \in \chi_t^-} \mathcal{V}_{(x,t)}^p \cap \mathcal{V}_{(y,t)}^p \cap ([-A, A] \times [t, r]) \quad (3.3)$$

$$\cup \bigcup_{x \in \chi_t^+ \cup \chi_t^-, y \in \chi_t^0} \mathcal{V}_{(x,t)}^p \cap (\{y\} \times [t, r]). \quad (3.4)$$

The set (3.3) is the possible locations (y, s) where two fires may meet during $[t, r]$. The set (3.4) is the possible locations (y, s) where a fire may be stopped by a microscopic zone or a barrier during $[t, r]$. Thus, \mathcal{E}_t^r is the set of possible locations (y, s) where a fire may be stopped during $[t, r]$, when no match falls in $[-A, A]$ during $[t, r]$.

Step 0. Put $Z_0^A(x) = H_0^A(x) = F_0^A(x) = 0$ for all $x \in [-A, A]$.

Assume that, for some $q \in \{0, \dots, n-1\}$, the process $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t \in [0, T_q], x \in [-A, A]}$ has been built.

Step $q+1$. We build $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t \in (T_q, T_{q+1}], x \in [-A, A]}$ in the following way: for $x \in [-A, A]$ and $t \in (T_q, T_{q+1})$, we set $H_t^A(x) = \max(0, H_{T_q}^A(x) - (t - T_q))$. We then set, recall (3.3) and (3.4),

$$\mathcal{E}_{T_q}^{T_{q+1}} = \{(X_q^1, T_q^1), \dots, (X_q^N, T_q^N)\}$$

ordered chronologically, and put $(X_q^0, T_q^0) = (X_q, T_q)$ and $(X_q^{N+1}, T_q^{N+1}) = (X_{q+1}, T_{q+1})$. Observe that a.s. $T_q = T_q^0 < T_q^1 < \dots < T_q^N < T_q^{N+1} = T_{q+1}$. Assume that the process has been built until T_q^k , for some $k \in \{0, \dots, N\}$. We then build the process on $(T_q^k, T_q^{k+1}]$. Recall that no match falls in $[-A, A]$ during the time interval (T_q^k, T_q^{k+1}) .

We first compute $(F_t^A(x))_{t \in (T_q^k, T_q^{k+1}), x \in [-A, A]}$. Since a fire can't be stopped during (T_q^k, T_q^{k+1}) , if $x \in \chi_{T_q^k}^+$, we set $F_s^A(y) = 1$ for all $(y, s) \in \mathcal{V}_{(x, T_q^k)}^p(x + \frac{T_q^{k+1} - T_q^k}{p}, T_q^{k+1})$, recall Subsection 2.1, while, if $x \in \chi_{T_q^k}^-$, we set $F_s^A(y) = 1$ for all $(y, s) \in \mathcal{V}_{(x, T_q^k)}^p(x - \frac{T_q^{k+1} - T_q^k}{p}, T_q^{k+1})$. Otherwise, that is if $(y, s) \notin \left(\bigcup_{x \in \chi_{T_q^k}^+} \mathcal{V}_{(x, T_q^k)}^p(x + \frac{T_q^{k+1} - T_q^k}{p}, T_q^{k+1}) \right) \cup \left(\bigcup_{x \in \chi_{T_q^k}^-} \mathcal{V}_{(x, T_q^k)}^p(x - \frac{T_q^{k+1} - T_q^k}{p}, T_q^{k+1}) \right)$, we set $F_s^A(y) = 0$. To summarize, for all $(y, s) \in [-A, A] \times (T_q^k, T_q^{k+1})$, we have

$$F_s^A(y) = \begin{cases} 1 & \text{if } y - \frac{s - T_q^k}{p} \in \chi_{T_q^k}^+ \\ 1 & \text{if } y + \frac{s - T_q^k}{p} \in \chi_{T_q^k}^- \\ 0 & \text{else.} \end{cases}$$

We then compute $(Z_t^A(x))_{t \in (T_q^k, T_q^{k+1}), x \in [-A, A]}$. Let us fix $x \in [-A, A]$. We set $N_x := \#\{s \in (T_q^k, T_q^{k+1}) : F_s^A(x) = 1\}$ and $\tau_0 := T_q^k$. If $N_x \geq 1$, for $j = 0, \dots, N_x - 1$, we set $\tau_{j+1} := \inf\{s \in (T_q^k, T_q^{k+1}) : F_s^A(x) = 1\}$. While x isn't crossed by a fire, $Z_s^A(x)$ grows linearly. We thus have, for all $s \in (T_q^k, T_q^{k+1})$

$$Z_s^A(x) = \begin{cases} \min(Z_{T_q^k}^A(x) + s - T_q^k, 1) & \text{if } s \in (T_q^k, \tau_1), \\ \min(s - \tau_j, 1) & \text{if } s \in [\tau_j, \tau_{j+1}) \text{ and } N_x \geq j \geq 1, \\ \min(s - \tau_{N_x}, 1) & \text{if } s \in [\tau_{N_x}, T_q^{k+1}). \end{cases}$$

if $N_x \geq 1$, whereas

$$Z_s^A(x) = \min(Z_{T_q^k}^A(x) + s - T_q^k, 1)$$

if $N_x = 0$.

We finally compute $F_{T_q^{k+1}}^A(x)$, $Z_{T_q^{k+1}}^A(x)$ and $H_{T_q^{k+1}}^A(x)$ for all $x \in [-A, A]$.

Case 1. If $x \neq X_q^{k+1}$, observe that at most one fire can reach x at time T_q^{k+1} (else $x \in \mathcal{E}_{T_q^k}^{T_q^{k+1}}$). If $x - \frac{T_q^{k+1} - T_q^k}{p} \in \chi_{T_q^k}^+$ or $x + \frac{T_q^{k+1} - T_q^k}{p} \in \chi_{T_q^k}^-$, that is if a fire reaches x at time T_q^{k+1} , we set $F_{T_q^{k+1}}^A(x) = 1$ and $Z_{T_q^{k+1}}^A(x) = 0$. Else, we set $F_{T_q^{k+1}}^A(x) = 0$ and $Z_{T_q^{k+1}}^A(x) = Z_{T_q^{k+1}-}^A(x)$.

Case 2. If $x = X_q^{k+1}$ and $k < N$, observe that X_q^{k+1} isn't crossed by a fire during (T_q^k, T_q^{k+1}) i.e. $N_{X_q^{k+1}} = 0$. If $X_q^{k+1} - \frac{T_q^{k+1} - T_q^k}{p} \notin \chi_{T_q^k}^+$ and $X_q^{k+1} + \frac{T_q^{k+1} - T_q^k}{p} \notin \chi_{T_q^k}^-$ (i.e. if the fire which might have reached X_q^{k+1} has been stopped before T_q^k) or if $H_{T_q^{k+1}-}^A(X_q^{k+1}) > 0$ or $Z_{T_q^{k+1}-}^A(X_q^{k+1}) < 1$ (i.e. if there has been recently a microscopic fire), then put $F_{T_q^{k+1}}^A(X_q^{k+1}) = 0$. Else, there is one (or two) fire that reaches X_q^{k+1} at time T_q^{k+1} and we set $F_{T_q^{k+1}}^A(X_q^{k+1}) = 1$ (or 2). To summarize, we put

$$F_{T_q^{k+1}}^A(X_q^{k+1}) = \mathbf{1}_{\{H_{T_q^{k+1}-}^A(X_q^{k+1})=0 \text{ and } Z_{T_q^{k+1}-}^A(X_q^{k+1})=1\}} \times \left(\mathbf{1}_{\{X_q^{k+1} - \frac{T_q^{k+1} - T_q^k}{p} \in \chi_{T_q^k}^+\}} + \mathbf{1}_{\{X_q^{k+1} + \frac{T_q^{k+1} - T_q^k}{p} \in \chi_{T_q^k}^-\}} \right).$$

We finally put

$$Z_{T_q^{k+1}}^A(X_q^{k+1}) = Z_{T_q^{k+1}-}^A(X_q^{k+1}) \mathbf{1}_{\{F_{T_q^{k+1}}^A(X_q^{k+1})=0\}}.$$

Case 3. If $x = X_{q+1} = X_q^{N+1}$ and $k = N$, a match falls in X_{q+1} at time $T_{q+1} = T_q^{N+1}$. We then set

$$Z_{T_{q+1}}^A(X_{q+1}) = Z_{T_{q+1}-}^A(X_{q+1}) \mathbf{1}_{\{Z_{T_{q+1}-}^A(X_{q+1}) < 1\}}$$

and

$$F_{T_{q+1}}^A(X_{q+1}) = \mathbf{1}_{\{Z_{T_{q+1}-}^A(X_{q+1})=1\}}.$$

To conclude the construction, we set, for all $x \in [-A, A]$

$$H_{T_{q+1}}^A(x) = \begin{cases} H_{T_{q+1}-}^A(x) & \text{if } x \neq X_{q+1}, \\ Z_{T_{q+1}-}^A(X_{q+1}) \mathbf{1}_{\{Z_{T_{q+1}-}^A(X_{q+1}) < 1\}} & \text{if } x = X_{q+1}. \quad \square \end{cases}$$

3.2.2 Restriction of the LFFP(p) to a finite box

We now prove a refined version of Theorem 2.4.

Proposition 3.5. *Let $p \in [0, \infty)$ and π_M be a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$.*

1. *There exists a unique LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$.*
2. *It can be perfectly simulated on $[-n, n] \times [0, T]$ for any $T > 0$, any $n > 0$.*
3. *For $A > 0$, let $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t \geq 0, x \in [-A, A]}$ be the unique A -LFFP(p) and the associated $(D_t^A(x))_{t \geq 0, x \in [-A, A]}$. There holds*

$$\begin{aligned} \mathbb{P} \left[(Z_t(x), H_t(x), F_t(x), D_t(x))_{t \in [0, T], x \in [-A/2, A/2]} \right. \\ \left. = (Z_t^A(x), H_t^A(x), F_t^A(x), D_t^A(x))_{t \in [0, T], x \in [-A/2, A/2]} \right] \geq 1 - C_T e^{-\alpha_T A} \end{aligned} \quad (3.5)$$

for some constants $\alpha_T > 0$ and $C_T > 0$ not depending on $A > 0$.

Proof. We divide the proof into several step. We work on $[0, T]$.

Step 1. We observe that for a mark (X, τ) of π_M with $X \in [-A, A]$, we have $H_t^A(X) > 0$ or $Z_t^A(X) < 1$ for all $t \in [\tau, \tau + 1/2]$.

Indeed, assume first that $Z_{\tau-}^A(X) \in [0, 1/2]$. Then $Z_t^A(X) = Z_{\tau-}^A(X) + t - \tau < 1$ for all $t \in [\tau, \tau + 1/2]$.

Assume next that $Z_{\tau-}^A(X) \in [1/2, 1]$. Then $H_{\tau}^A(X) = Z_{\tau-}^A \geq 1/2$, so that $H_t^A(X) = H_{\tau}^A(X) - t + \tau > 0$ for all $t \in [\tau, \tau + 1/2]$.

If finally $Z_{\tau-}^A(X) = 1$, then $Z_{\tau}^A(X) = 0$, whence $Z_t^A(X) = t - \tau < 1$ for $t \in [\tau, \tau + 1]$.

Step 2. For $a \in \mathbb{R}$, we consider the event Ω_a^l defined as follows: for $\{(X_k, T_k)\}_{k=1, \dots, n}$ the marks of π_M restricted to $[a, a+1) \times [0, T]$ ordered chronologically, for $T_0 = 0, T_{n+1} = T$, we put $\Omega_a^l = \{\max_{i=0, \dots, n}(T_{i+1} - T_i) < 1/4\} \cap \{\min_{i=1, \dots, n-1}(X_{i+1} - X_i) > 0\}$.

We immediately deduce from Step 1 that for any $a \in \mathbb{R}$, any $A > |a| + 1$,

$$\Omega_a^l \subset \{\exists x : [0, T] \rightarrow (a, a+1), t \mapsto x_t \text{ non decreasing}$$

$$\text{and for all } t \in [0, T], H_t^A(x_t) > 0 \text{ or } Z_t^A(x_t) < 1\}.$$

Thus, on Ω_a^l , clusters on the left of a cannot be connected to clusters on the right of $a+1$ during $[0, T]$. Furthermore, since the function x is non decreasing, a fire starting from the left of a can't cross the zone $(a, a+1)$ (i.e. it necessarily would be stopped by some x_{t_0}). Thus, matches falling at the left of a do not affect the zone $(a+1, \infty)$.

In the same way, we put $\Omega_a^r = \{\max_{i=0, \dots, n}(T_{i+1} - T_i) < 1/4\} \cap \{\max_{i=1, \dots, n-1}(X_{i+1} - X_i) < 0\}$. We of course have, for any $a \in \mathbb{R}, A > |a| + 1$,

$$\Omega_a^r \subset \{\exists y : [0, T] \rightarrow (a, a+1), t \mapsto y_t \text{ non increasing}$$

$$\text{and for all } t \in [0, T], H_t^A(y_t) > 0 \text{ or } Z_t^A(y_t) < 1\}.$$

As above, on Ω_a^r , clusters on the right of $a+1$ cannot be connected to clusters on the left of a during $[0, T]$ and the fact that y is non increasing ensures us that matches falling on the right on $a+1$ do not affect the zone $(-\infty, a)$.

Step 3. Obviously, $q_T = \mathbb{P}[\Omega_a^l] = \mathbb{P}[\Omega_a^r]$ is positive and does not depend on a . Furthermore, Ω_a^l (resp. Ω_a^r) is independent of Ω_b^l (resp. Ω_b^r) for all $a, b \in \mathbb{Z}$ with $a \neq b$. Hence there are a.s. infinitely many $a \in \mathbb{Z}$ (resp. $b \in \mathbb{Z}$) such that Ω_a^l (resp. Ω_b^r) is realized.

Then it is routine to deduce the well-posedness of the LFFP(p). The perfect simulation algorithm on a finite-box $[-n, n] \times [0, T]$ is also easy: find $a_1 < a_2$ with $a_1 + 1 < -n < n < a_2$ such that $\Omega_{a_1}^l \cap \Omega_{a_2}^r$ is realized. Then apply the same rules as for the A -LFFP(p) to simulate the process in $[a_1, a_2 + 1]$. This will give the true LFFP(p) inside $[a_1 + 1, a_2]$ during $[0, T]$.

Finally, we can clearly bound from below the left hand side of (3.5) by

$$\mathbb{P}[(\cup_{a \in [-A, -A/2-1] \cap \mathbb{Z}} \Omega_a^l) \cap (\cup_{a \in [A/2, A-1] \cap \mathbb{Z}} \Omega_a^r)] \geq 1 - 2(1 - q_T)^{A/2-2}$$

whence (3.5) with $C_T = 2/(1 - q_T)^2$ and $\alpha_T = -\log(1 - q_T)/2$. \square

4 Propagation Lemmas

Here we study the propagation of a fire through an occupied cluster. When a match falls on an occupied cluster, two fires start: one goes to the left and one goes to the right. This propagation is not necessarily linear, it sometimes can regress. However there are few 'sparks'.

Consider two families of Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all these processes being independent. We consider *the propagation process ignited*

at $(0, 0)$ defined by

$$\begin{aligned}\check{\zeta}_t^{\lambda, \pi}(i) &= 1 + \mathbf{1}_{\{i=0\}} + \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda, \pi}(i)=0\}} dN_s^S(i) \\ &+ \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda, \pi}(i+1)=2, \check{\zeta}_{s-}^{\lambda, \pi}(i)=1\}} dN_s^P(i+1) + \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda, \pi}(i-1)=2, \check{\zeta}_{s-}^{\lambda, \pi}(i)=1\}} dN_s^P(i-1) \\ &- 2 \int_0^t \mathbf{1}_{\{\check{\zeta}_{s-}^{\lambda, \pi}(i)=2\}} dN_s^P(i).\end{aligned}$$

Roughly, the process $(\check{\zeta}_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ starts from an occupied initial configuration and a match falls on the site 0 at time 0. Afterwards the fire spreads into \mathbb{Z} . We are interested in the space-time position of burning trees (i.e. $(i, t) \in \mathbb{Z} \times [0, \infty)$ such that $\check{\zeta}_t^{\lambda, \pi}(i) = 2$), when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the different regimes.

We set, for $t \geq 0$,

$$i_t^+ = \max \left\{ i \geq 0 : \check{\zeta}_t^{\lambda, \pi}(i) = 2 \right\} \quad (4.1)$$

$$i_t^- = \min \left\{ i \leq 0 : \check{\zeta}_t^{\lambda, \pi}(i) = 2 \right\} \quad (4.2)$$

the right and the left fronts at time t . Observe that $(i_t^+)_{t \geq 0}$ and $(-i_t^-)_{t \geq 0}$ are two Poisson processes with intensity π . For $i \in \mathbb{Z}$, we set

$$\begin{aligned}T_i &= \inf \left\{ s \geq 0 : \check{\zeta}_s^{\lambda, \pi}(i) = 2 \right\} \\ &= \begin{cases} \inf \{ s \geq 0 : i_s^+ = i \} & \text{if } i \geq 0, \\ \inf \{ s \geq 0 : i_s^- = i \} & \text{if } i \leq 0, \end{cases}\end{aligned} \quad (4.3)$$

which represents the first time that the site $i \in \mathbb{N}$ is burning. We clearly have for all $t \geq 0$,

$$\check{\zeta}_t^{\lambda, \pi}(i_t^-) = 2 = \check{\zeta}_t^{\lambda, \pi}(i_t^+)$$

and for all $i \notin \llbracket i_t^-, i_t^+ \rrbracket$,

$$\check{\zeta}_t^{\lambda, \pi}(i) = 1.$$

In this section, we will show that burning trees at some time t are *concentrated* around i_t^+ and i_t^- . We say that a site i is a *spark* at time t if it is a burning tree such that $i \notin \{i_t^-, i_t^+\}$.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \left\lfloor \frac{1}{\lambda \mathbf{a}_\lambda} \right\rfloor$ and we introduce $\varepsilon_\lambda = \frac{1}{\mathbf{a}_\lambda^3}$. For $B > 0$, we finally set $B_\lambda = \lfloor B \mathbf{n}_\lambda \rfloor$.

The following Definition will be useful.

Definition 4.1. *Let $p \geq 0$. In the rest of the paper, we will say that a statement $\mathcal{S}(\lambda, \pi)$ holds for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$ if there are $\varepsilon_0 > 0$ and $\lambda_0 \in (0, 1)$ such that for all $\lambda \in (0, \lambda_0)$ and all $\pi \geq 1$ such that $\left| \frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p \right| < \varepsilon_0$, the statement $\mathcal{S}(\lambda, \pi)$ holds.*

Similarly, let $z_0 \in [0, 1]$. We will say that a statement $\mathcal{S}(\lambda, \pi)$ holds for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$ if there are $\varepsilon_0 > 0$, $\lambda_0 \in (0, 1)$ and $K_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and all $\pi \geq 1$ such that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} > K_0$ and $\left| \frac{\log(\pi)}{\log(1/\lambda)} - z_0 \right| < \varepsilon_0$, the statement $\mathcal{S}(\lambda, \pi)$ holds.

4.1 Propagation lemma in the regime $\mathcal{R}(p)$, for some $p \in (0, \infty)$

We first study the propagation in the regime $\mathcal{R}(p)$, for some $p > 0$.

Lemma 4.2. *Let $p > 0, T > 0$. There exists an event $\Omega_{\lambda, \pi}^{P, T}$ depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t \in [0, \mathbf{a}_\lambda(T + \varepsilon_\lambda)], i \in \llbracket -\lfloor \mathbf{a}_\lambda \pi(T + \varepsilon_\lambda) \rfloor, \lfloor \mathbf{a}_\lambda \pi(T + \varepsilon_\lambda) \rfloor \rrbracket}$ such that*

$$\Omega_{\lambda, \pi}^{P, T} \subset \{ \text{At any time } t \in [0, \mathbf{a}_\lambda T], \text{ any burning tree belongs to}$$

$$\llbracket -\lfloor (t + \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor, -\lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor \rrbracket \cup \llbracket \lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor, \lfloor (t + \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor \rrbracket$$

and is either i_t^+ or i_t^- or has vacant neighbors $\}$,

where the event on the right concerns $(\check{\zeta}_t^{\lambda,\pi}(i))_{i \in \mathbb{Z}, t \geq 0}$, and

$$\lim_{\lambda, \pi} \mathbb{P} \left[\Omega_{\lambda, \pi}^{P, T} \right] = 1$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

Proof. Recall that a *spark* at time t is a burning tree i such that $i \notin \{i_t^-, i_t^+\}$. We say that a site i *propagates for the first time* when the first fire at i extinguishes and spreads to its neighbors (if they are occupied). Observe that for $i \geq 0$, this happens at time T_{i+1} , while for $i \leq 0$, this happens at time T_{i-1} .

Consider, for $i \geq 0$, the events

$$\Omega_i^1 = \{i \text{ remains vacant from the instant at which it propagates for the first time} \\ \text{until the instant at which the fire in } i+1 \text{ propagates for the first time}\} \quad (4.4)$$

and

$$\Omega_i^2 = \{i \text{ is occupied when the fire in } i+1 \text{ propagates for the first time,} \\ \text{but then, } i \text{ burns for the second time during less than } \mathbf{a}_\lambda \varepsilon_\lambda / 4 \\ \text{and no seed has fallen on its neighbors } i-1, i+1 \\ \text{from the instant they burnt for the first time until } i \text{ propagates for the second time}\} \quad (4.5)$$

and similar events for $i \leq 0$ (replace $i+1$ by $i-1$). Recall (4.1), (4.2) and remark that the event on the right hand side in Lemma 4.2 contains the event

$$\Omega_{\lambda, \pi}^{P, T} = \left\{ \sup_{t \in [0, \mathbf{a}_\lambda T]} |i_t^+ - \pi t| \leq \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2} \right\} \cap \left\{ \sup_{t \in [0, \mathbf{a}_\lambda T]} |i_t^- + \pi t| \leq \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2} \right\} \\ \cap \{ \forall i \in \llbracket i_{\mathbf{a}_\lambda T}^- + 1, i_{\mathbf{a}_\lambda T}^+ - 1 \rrbracket, \Omega_i^1 \text{ or } \Omega_i^2 \text{ is realized} \}.$$

Indeed, the two first terms ensure that the right (resp. left) front at time $t \in [0, \mathbf{a}_\lambda T]$ belongs to $\llbracket \lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda / 2) \pi \rfloor, \lfloor (t + \mathbf{a}_\lambda \varepsilon_\lambda / 2) \pi \rfloor \rrbracket$ (resp. $\llbracket -\lfloor (t + \mathbf{a}_\lambda \varepsilon_\lambda / 2) \pi \rfloor, -\lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda / 2) \pi \rfloor \rrbracket$). This in particular implies that for all $i \in \llbracket -\lfloor (T - \varepsilon_\lambda / 2) \mathbf{a}_\lambda \pi \rfloor, \lfloor (T - \varepsilon_\lambda / 2) \mathbf{a}_\lambda \pi \rfloor \rrbracket$,

$$T_i \in \left[\frac{|i|}{\pi} - \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{2}, \frac{|i|}{\pi} + \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{2} \right].$$

The last term says that either i remains vacant until $i+1$ propagates (i.e. there is no spark) or a seed has fallen on i but then i has vacant neighbors when it propagates for the second time (i.e. the spark has a size 1). Finally remark that on $\Omega_{\lambda, \pi}^{P, T}$, for $t \in [0, \mathbf{a}_\lambda T]$,

$$\left\{ 0 \leq i \leq i_t^+ : T_{i+2} + \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4} \geq t \right\} \subset \llbracket \lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor, i_t^+ \rrbracket$$

and

$$\left\{ 0 \geq i \geq i_t^- : T_{i-2} + \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4} \geq t \right\} \subset \llbracket i_t^-, -\lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor \rrbracket,$$

thus a burning tree (i.e. a front or a spark) necessarily belongs to $\llbracket \lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor, \lfloor (t + \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor \rrbracket \cup \llbracket -\lfloor (t + \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor, -\lfloor (t - \mathbf{a}_\lambda \varepsilon_\lambda) \pi \rfloor \rrbracket$, as desired.

Clearly, $\Omega_{\lambda, \pi}^{P, T}$ depends only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ through $t \in [0, \mathbf{a}_\lambda(T + \varepsilon_\lambda)]$ and $i \in \llbracket -\lfloor \mathbf{a}_\lambda \pi(T + \varepsilon_\lambda) \rfloor, \lfloor \mathbf{a}_\lambda \pi(T + \varepsilon_\lambda) \rfloor \rrbracket$. It remains to prove that $\mathbb{P} \left[\Omega_{\lambda, \pi}^{P, T} \right]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

Since $(i_t^+)_{t \geq 0}$ and $(-i_t^-)_{t \geq 0}$ are two Poisson processes with intensity π , the maximal inequality for martingales gives

$$\begin{aligned} \mathbb{P} \left[\sup_{t \in [0, \mathbf{a}_\lambda T]} |i_t^- + \pi t| > \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2} \right] &= \mathbb{P} \left[\sup_{t \in [0, \mathbf{a}_\lambda T]} |i_t^+ - \pi t| > \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2} \right] \\ &\leq \left(\frac{2}{\mathbf{a}_\lambda \pi \varepsilon_\lambda} \right)^4 \times (3(\mathbf{a}_\lambda \pi T)^2 + \mathbf{a}_\lambda \pi T) \\ &\leq \frac{16T^2}{(\mathbf{a}_\lambda \pi \varepsilon_\lambda^2)^2} = \frac{16T^2 \mathbf{a}_\lambda^{10}}{\pi^2} \end{aligned} \quad (4.6)$$

which tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

Next, for all $i \geq 0$, we have

$$\mathbb{P} [\Omega_i^1] = \frac{\pi}{1 + \pi} \quad (4.7)$$

because seeds fall on i at rate 1 while the fire on $i + 1$ propagates at rate π .

Now, for all $i \geq 0$, we set

$$\begin{aligned} X_i &= \inf \left\{ s > T_{i+1} : N_s^S(i) - N_{T_{i+1}}^S(i) > 0 \right\} - T_{i+1}, \\ Y_i^1 &= T_{i+1} - T_i, \\ Y_i^2 &= \inf \left\{ s > T_{i+2} : N_s^P(i) - N_{T_{i+2}}^P(i) > 0 \right\} - T_{i+2}. \end{aligned}$$

Let $i \geq 0$. At time T_i , the site i is burning and propagates to neighbors at time T_{i+1} . Thus, X_i is the time we have to wait for a seed to fall again on i after it propagates for the first time. Furthermore, Y_i^1 stands for the duration that i is burning for the first time. If a seed falls on i before T_{i+2} , that is before the burning tree $i + 1$ propagates, then i becomes again burning at time T_{i+2} and burns during $[T_{i+2}, T_{i+2} + Y_i^2]$.

The random variables $(X_i)_{i \in \mathbb{N}}$ are exponential random variables with parameter 1 and the random variables $(Y_i^1)_{i \in \mathbb{N}}$ and $(Y_i^2)_{i \in \mathbb{N}}$ are exponential random variables with parameter π . All these random variables are independent.

Then observe that

$$\Omega_i^2 = \left(\{X_i \leq Y_{i+1}^1\} \cap \{Y_i^2 < \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4}\} \cap \{X_{i-1} > Y_i^1 + Y_{i+1}^1 + Y_i^2\} \cap \{X_{i+1} > Y_i^2\} \right). \quad (4.8)$$

We have by independence

$$\begin{aligned} \mathbb{P} [\Omega_i^2 \mid Y_i^1, Y_{i+1}^1, Y_i^2] &= (1 - e^{-Y_{i+1}^1}) \times \mathbf{1}_{\{Y_i^2 \leq \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4}\}} \times e^{-(Y_i^1 + Y_{i+1}^1 + Y_i^2)} \times e^{-Y_i^2} \\ &= (1 - e^{-Y_{i+1}^1}) \times e^{-Y_{i+1}^1} \times e^{-Y_i^1} \times e^{-2Y_i^2} \times \mathbf{1}_{\{Y_i^2 \leq \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{4}\}}. \end{aligned}$$

Integrating,

$$\begin{aligned} \mathbb{P} [\Omega_i^2] &= \pi^3 \int_0^\infty (1 - e^{-x}) e^{-(\pi+1)x} dx \times \int_0^\infty e^{-(\pi+1)y} dy \times \int_0^{\mathbf{a}_\lambda \varepsilon_\lambda / 4} e^{-(\pi+2)z} dz \\ &= \frac{\pi^3}{(1 + \pi)^2 (2 + \pi)^2} (1 - e^{-(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda / 4}). \end{aligned} \quad (4.9)$$

Finally, note that, in the regime $\mathcal{R}(p)$,

$$\begin{aligned} \mathbb{P} [\Omega_i^1 \cup \Omega_i^2] &= \mathbb{P} [\Omega_i^1] + \mathbb{P} [\Omega_i^2] = \frac{\pi}{1 + \pi} + \frac{\pi^3}{(1 + \pi)^2 (2 + \pi)^2} (1 - e^{-(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda / 4}) \\ &= 1 - \frac{5\pi^2 + 8\pi + 4 + \pi^3 e^{-(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda / 4}}{(1 + \pi)^2 (2 + \pi)^2} \\ &\geq 1 - \frac{\alpha}{\pi^2} \end{aligned}$$

for some constant $\alpha > 0$, because $e^{-(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda / 4} \ll 1/\pi$ when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$ (indeed, $\pi \sim 1/(p\lambda \log^2(1/\lambda))$ whence $(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda \simeq 1/(p\lambda \log^3(1/\lambda))$). Similar computations hold for $i \leq 0$.

Consequently, the probability of $\{\forall i \in \llbracket i_{\mathbf{a}_\lambda T}^- + 1, i_{\mathbf{a}_\lambda T}^+ - 1 \rrbracket, \Omega_i^1 \text{ or } \Omega_i^2 \text{ is realized}\}$ knowing $\left\{ \sup_{t \in [0, \mathbf{a}_\lambda T]} |i_t^+ - \pi t| \leq \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2} \right\} \cap \left\{ \sup_{t \in [0, \mathbf{a}_\lambda T]} |i_t^- + \pi t| \leq \frac{\mathbf{a}_\lambda \pi \varepsilon_\lambda}{2} \right\}$ is bounded from below by

$$\begin{aligned} 1 - \sum_{i = -\lfloor \mathbf{a}_\lambda \pi (T + \varepsilon_\lambda) \rfloor}^{\lfloor \mathbf{a}_\lambda \pi (T + \varepsilon_\lambda) \rfloor} \mathbb{P}[(\Omega_i^1 \cup \Omega_i^2)^c] &= 1 - \sum_{i = -\lfloor \mathbf{a}_\lambda \pi (T + \varepsilon_\lambda) \rfloor}^{\lfloor \mathbf{a}_\lambda \pi (T + \varepsilon_\lambda) \rfloor} (1 - \mathbb{P}[\Omega_i^1] - \mathbb{P}[\Omega_i^2]) \\ &\geq 1 - \alpha \frac{\mathbf{a}_\lambda \pi (T + 1)}{\pi^2} = 1 - \alpha_T \frac{\mathbf{a}_\lambda}{\pi} \end{aligned} \quad (4.10)$$

which tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$. Gathering (4.6) and (4.10) concludes the proof of Lemma 4.2. \square

4.2 Propagation lemma in the regime $\mathcal{R}(0)$

For all $A > 0$, we set

$$\varkappa_{\lambda, \pi}^A = \frac{\mathbf{n}_\lambda A}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda \quad (4.11)$$

which tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

Lemma 4.3. *Let $A, B > 0$. There exists an event $\Omega_{\lambda, \pi}^{P, A, B}$ depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t \in [0, \mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B}], i \in \llbracket -A_\lambda - \mathbf{m}_\lambda, B_\lambda + \mathbf{m}_\lambda \rrbracket}$ such that*

$$\begin{aligned} \Omega_{\lambda, \pi}^{P, A, B} \subset \{ &\text{There is no more burning tree in } \llbracket -A_\lambda, B_\lambda \rrbracket \text{ at time } \mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B} \\ &\text{and a burning tree in } \llbracket -A_\lambda, B_\lambda \rrbracket \text{ at some time } 0 \leq t \leq \mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B} \\ &\text{is either } i_t^+ \text{ or } i_t^- \text{ or has vacant neighbors} \} \end{aligned}$$

where the event on the right concerns $(\check{\zeta}_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, and

$$\lim_{\lambda, \pi} \mathbb{P}[\Omega_{\lambda, \pi}^{P, A, B}] = 1$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

Proof. Recall (4.3), (4.4) and (4.5). We set

$$\begin{aligned} \Omega_{\lambda, \pi}^{P, A, B} &= \left\{ T_{B_\lambda + \mathbf{m}_\lambda} \leq \frac{\mathbf{n}_\lambda B}{\pi} + \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{2} \right\} \cap \left\{ T_{-A_\lambda - \mathbf{m}_\lambda} \leq \frac{\mathbf{n}_\lambda A}{\pi} + \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{2} \right\} \\ &\quad \cap \bigcap_{i \in \llbracket -A_\lambda - \mathbf{m}_\lambda + 1, B_\lambda + \mathbf{m}_\lambda - 1 \rrbracket} (\Omega_i^1 \cup \Omega_i^2) \\ &\cap \left\{ \exists i \in \llbracket -A_\lambda - \mathbf{m}_\lambda + 1, -A_\lambda \rrbracket, N_{\mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B}}^S(i) = 0 \right\} \cap \left\{ \exists i \in \llbracket B_\lambda, B_\lambda + \mathbf{m}_\lambda - 1 \rrbracket, N_{\mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B}}^S(i) = 0 \right\}. \end{aligned}$$

Observe now that the event on the right hand side in Lemma 4.3 contains the event $\Omega_{\lambda, \pi}^{P, A, B}$. Indeed, the two first terms ensure that the left and the right fronts are outside $\llbracket -A_\lambda, B_\lambda \rrbracket$ at time $\mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B}$ whereas the third term ensures that a spark burns not for a long time and has vacants neighbors. The two last terms prevent from a return of a fire.

It remains to prove that $\mathbb{P}[\Omega_{\lambda, \pi}^{P, A, B}]$ tends to 1. First, observe that $T_{B_\lambda + \mathbf{m}_\lambda}$ is a sum of $B_\lambda + \mathbf{m}_\lambda$ i.i.d. exponential random variables with parameter π , then, Chebyshev's inequality implies

$$\begin{aligned} \mathbb{P} \left[T_{B_\lambda + \mathbf{m}_\lambda} > \frac{\mathbf{n}_\lambda B}{\pi} + \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{2} \right] &\leq \mathbb{P} \left[\left| T_{B_\lambda + \mathbf{m}_\lambda} - \frac{\mathbf{n}_\lambda B}{\pi} \right| > \frac{\mathbf{a}_\lambda \varepsilon_\lambda}{2} \right] \leq \frac{4}{(\mathbf{a}_\lambda \varepsilon_\lambda)^2} \frac{B_\lambda + \mathbf{m}_\lambda}{\pi^2} \\ &\leq C_B \frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \frac{1}{\mathbf{a}_\lambda \pi \varepsilon_\lambda^2} \end{aligned}$$

which tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$. Similar computation holds for $T_{-A_\lambda - \mathbf{m}_\lambda}$.

A basic calculation, as in (4.10), shows that (because it also holds true that $e^{-(2+\pi)\mathbf{a}_\lambda \varepsilon_\lambda/4} \ll 1/\pi$ in the regime $\mathcal{R}(0)$)

$$\mathbb{P} \left[\bigcap_{i \in \llbracket -A_\lambda - \mathbf{m}_\lambda + 1, B_\lambda + \mathbf{m}_\lambda - 1 \rrbracket} (\Omega_i^1 \cup \Omega_i^2) \right] \geq 1 - \alpha_T \frac{\mathbf{a}_\lambda}{\pi} \quad (\text{for some } \alpha_T > 0),$$

which tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

Finally, as soon as $\varkappa_{\lambda, \pi}^{A \vee B} \leq \frac{1}{2}$, it holds that, using space stationarity,

$$\begin{aligned} \mathbb{P} \left[\exists i \in \llbracket B_\lambda, B_\lambda + \mathbf{m}_\lambda - 1 \rrbracket, N_{\mathbf{a}_\lambda \varkappa_{\lambda, \pi}^{A \vee B}}^S(i) = 0 \right] &\geq \mathbb{P} \left[\exists i \in \llbracket 0, \mathbf{m}_\lambda - 1 \rrbracket, N_{\mathbf{a}_\lambda/2}^S(i) = 0 \right] \\ &= 1 - (1 - e^{-\mathbf{a}_\lambda/2})^{\mathbf{m}_\lambda - 1} \simeq 1 - e^{-\sqrt{\lambda}(\mathbf{m}_\lambda - 1)} \end{aligned}$$

which tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$. \square

4.3 Propagation lemma in the regime $\mathcal{R}(\infty, z_0)$

We first introduce, for $\lambda \in (0, 1]$ and $\gamma \in (0, 1)$,

$$\mathbf{m}_\lambda^\gamma = \left\lfloor \frac{\gamma}{\lambda^{\gamma + (1-\gamma)z_0} \mathbf{a}_\lambda} \right\rfloor.$$

For $z_0 = 1$, $\mathbf{m}_\lambda^\gamma$ is nothing but $\lfloor \gamma \mathbf{n}_\lambda \rfloor$. For $z_0 \in [0, 1)$ and $\gamma \in (0, 1)$, observe that

$$z_0 < \gamma + (1 - \gamma)z_0 < 1,$$

so that $\mathbf{m}_\lambda^\gamma \ll \mathbf{n}_\lambda$. In any cases, we have $\mathbf{m}_\lambda^\gamma / \mathbf{n}_\lambda \leq \gamma$.

Lemma 4.4. *Let $T > 0$. For all $z_0 \in [0, 1]$ and all $\gamma \in (0, 1)$, there exists an event $\Omega_{\lambda, \pi}^{P, T, \gamma}$ depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t \in [0, \mathbf{a}_\lambda T], i \in \llbracket -\mathbf{m}_\lambda^\gamma, \mathbf{m}_\lambda^\gamma \rrbracket}$, such that*

$$\Omega_{\lambda, \pi}^{P, T, \gamma} \subset \{i_{\mathbf{a}_\lambda T}^+ \text{ and } i_{\mathbf{a}_\lambda T}^- \text{ belong to } \llbracket -\mathbf{m}_\lambda^\gamma, \mathbf{m}_\lambda^\gamma \rrbracket\},$$

where the event on the right concerns the process $(\check{\zeta}_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, and

$$\lim_{\lambda, \pi} \mathbb{P} \left[\Omega_{\lambda, \pi}^{P, T, \gamma} \right] = 1$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Proof. Recall (4.1) and (4.2). We define

$$\Omega_{\lambda, \pi}^{P, T, \gamma} = \{0 \leq i_{\mathbf{a}_\lambda T}^+ \leq \mathbf{m}_\lambda^\gamma\} \cap \{-\mathbf{m}_\lambda^\gamma \leq i_{\mathbf{a}_\lambda T}^- \leq 0\},$$

which clearly implies that $i_{\mathbf{a}_\lambda T}^+$ and $i_{\mathbf{a}_\lambda T}^-$ belong to $\llbracket -\mathbf{m}_\lambda^\gamma, \mathbf{m}_\lambda^\gamma \rrbracket$. Markov's inequality shows that

$$\mathbb{P} \left[i_{\mathbf{a}_\lambda T}^- < -\mathbf{m}_\lambda^\gamma \right] = \mathbb{P} \left[i_{\mathbf{a}_\lambda T}^+ > \mathbf{m}_\lambda^\gamma \right] \leq \frac{\mathbf{a}_\lambda \pi T}{\mathbf{m}_\lambda^\gamma} \simeq \frac{T}{\gamma} \mathbf{a}_\lambda^2 \pi \lambda^{\gamma + (1-\gamma)z_0},$$

which tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$. Indeed, for $z_0 = 1$, then $\frac{T}{\gamma} \mathbf{a}_\lambda^2 \pi \lambda = \frac{T}{\gamma} \frac{\mathbf{a}_\lambda \pi}{\mathbf{n}_\lambda}$ tends to 0 (it is the definition of the regime $\mathcal{R}(\infty, 1)$), while, for $z_0 \in [0, 1)$, $z_0 < \gamma + (1 - \gamma)z_0 < 1$, then $\frac{T}{\gamma} \mathbf{a}_\lambda^2 \pi \lambda^{\gamma + (1-\gamma)z_0} = \frac{T}{\gamma} \frac{\mathbf{a}_\lambda^2 \pi}{\lambda^{z_0}} \lambda^{(1-z_0)\gamma}$ tends to 0, because $\log(\pi)/\log(1/\lambda)$ tends to z_0 . \square

For $z \in (0, 1)$, we next define

$$\kappa_{\lambda, \pi}^z = \frac{1}{\lambda^z \mathbf{a}_\lambda \pi} + \varepsilon_\lambda.$$

Observe that, if $0 < z < z_0$, then $\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z$ tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Lemma 4.5. For all $z_0 \in (0, 1]$ and all $z \in (0, z_0)$, there exists an event $\Omega_{\lambda, \pi}^{P, z}$, depending only on the Poisson processes $(N_t^S(i), N_t^P(i))_{t \in [0, \mathbf{a}_\lambda T], i \in [-\mathbf{m}_\lambda^\gamma, \mathbf{m}_\lambda^\gamma]}$, such that

$$\Omega_{\lambda, \pi}^{P, z} \subset \{i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+ \text{ and } -i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^- \text{ are greater than } \lfloor \lambda^{-z} \rfloor\}$$

$$\text{and all } i \in \llbracket i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^- + 1, i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+ - 1 \rrbracket \text{ burns exactly once before } \mathbf{a}_\lambda \kappa_{\lambda, \pi}^z \},$$

where the event on the right concerns the process $(\check{\zeta}_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, and

$$\lim_{\lambda, \pi} \mathbb{P} \left[\Omega_{\lambda, \pi}^{P, z} \right] = 1$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Proof. Let $z \in (0, z_0)$. Recall (4.1), (4.2), (4.4) and remark that $\mathbf{a}_\lambda \pi \kappa_{\lambda, \pi}^z = \lambda^{-z} + \mathbf{a}_\lambda \pi \varepsilon_\lambda$. We define

$$\Omega_{\lambda, \pi}^{P, z} = \left\{ i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+ \in \llbracket \lfloor \lambda^{-z} \rfloor, \lfloor \lambda^{-z} + 2\mathbf{a}_\lambda \pi \varepsilon_\lambda \rfloor \rrbracket \right\} \cap \left\{ i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^- \in \llbracket -\lfloor \lambda^{-z} - 2\mathbf{a}_\lambda \pi \varepsilon_\lambda \rfloor, \lfloor \lambda^{-z} \rfloor \rrbracket \right\}$$

$$\cap \bigcap_{i \in \llbracket -\lfloor \lambda^{-z} + 2\mathbf{a}_\lambda \pi \varepsilon_\lambda \rfloor, \lfloor \lambda^{-z} + 2\mathbf{a}_\lambda \pi \varepsilon_\lambda \rfloor \rrbracket} \Omega_i^1.$$

Observe that the event on the right hand side in Lemma 4.5 contains the event $\Omega_{\lambda, \pi}^{P, z}$. Indeed, as in the proof of Lemma 4.2, the two first terms situate the left and the right fronts. The third term ensures that there is no spark in the zone $\llbracket -\lfloor \mathbf{a}_\lambda \pi (\kappa_{\lambda, \pi}^z + \varepsilon_\lambda) \rfloor, \lfloor \mathbf{a}_\lambda \pi (\kappa_{\lambda, \pi}^z + \varepsilon_\lambda) \rfloor \rrbracket \supset \llbracket i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^-, i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+ \rrbracket \supset \llbracket -\lfloor \lambda^{-z} \rfloor, \lfloor \lambda^{-z} \rfloor \rrbracket$.

Since $i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+$ and $-i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^-$ are two Poisson random variables with parameter $\mathbf{a}_\lambda \pi \kappa_{\lambda, \pi}^z$, Chebychev's inequality shows

$$\mathbb{P} \left[i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^- \notin \llbracket -\lfloor \lambda^{-z} - 2\mathbf{a}_\lambda \pi \varepsilon_\lambda \rfloor, -\lfloor \lambda^{-z} \rfloor \rrbracket \right] = \mathbb{P} \left[|i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^- + \mathbf{a}_\lambda \pi \kappa_{\lambda, \pi}^z| > \mathbf{a}_\lambda \pi \varepsilon_\lambda \right]$$

$$= \mathbb{P} \left[i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+ \notin \llbracket \lfloor \lambda^{-z} \rfloor, \lfloor \lambda^{-z} + 2\mathbf{a}_\lambda \pi \varepsilon_\lambda \rfloor \rrbracket \right] = \mathbb{P} \left[\left| i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^z}^+ - \mathbf{a}_\lambda \pi \kappa_{\lambda, \pi}^z \right| > \mathbf{a}_\lambda \pi \varepsilon_\lambda \right]$$

$$\leq \frac{\mathbf{a}_\lambda \pi \kappa_{\lambda, \pi}^z}{(\mathbf{a}_\lambda \pi \varepsilon_\lambda)^2} = \frac{\kappa_{\lambda, \pi}^z}{\mathbf{a}_\lambda \pi \varepsilon_\lambda^2} = \kappa_{\lambda, \pi}^z \frac{\mathbf{a}_\lambda^3}{\pi}$$

which again tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$ (because $\log(\pi) \sim z_0 \mathbf{a}_\lambda$).

Finally, we still have $\mathbb{P}[\Omega_i^1] = \frac{\pi}{1+\pi}$, recall (4.7), whence

$$\mathbb{P} \left[\bigcap_{i \in \llbracket -\lfloor \mathbf{a}_\lambda \pi (\kappa_{\lambda, \pi}^z + \varepsilon_\lambda) \rfloor, \lfloor \mathbf{a}_\lambda \pi (\kappa_{\lambda, \pi}^z + \varepsilon_\lambda) \rfloor \rrbracket} \Omega_i^1 \right] = \left(\frac{\pi}{1+\pi} \right)^{2\lfloor \mathbf{a}_\lambda \pi (\kappa_{\lambda, \pi}^z + \varepsilon_\lambda) \rfloor + 1} \simeq e^{-2\mathbf{a}_\lambda (\kappa_{\lambda, \pi}^z + \varepsilon_\lambda)}$$

which tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$. This concludes the proof of Lemma 4.5. \square

4.4 Application to the (λ, π) -FFP

We next give some useful definitions.

Definition 4.6. Consider two families of Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all these processes being independent. Let $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$. We call

- propagation process ignited at (x_0, t_0) the process $(\check{\zeta}_t^{\lambda, \pi, 0}(i))_{t \geq 0, i \in \mathbb{Z}}$ built using the seed processes family $(N_t^{S, 0}(i))_{t \geq 0, i \in \mathbb{Z}} = (N_{t+\mathbf{a}_\lambda t_0}^S(i + \lfloor \mathbf{n}_\lambda x_0 \rfloor) - N_{\mathbf{a}_\lambda t_0}^S(i + \lfloor \mathbf{n}_\lambda x_0 \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^{P, 0}(i))_{t \geq 0, i \in \mathbb{Z}} = (N_{t+\mathbf{a}_\lambda t_0}^P(i + \lfloor \mathbf{n}_\lambda x_0 \rfloor) - N_{\mathbf{a}_\lambda t_0}^P(i + \lfloor \mathbf{n}_\lambda x_0 \rfloor))_{t \geq 0, i \in \mathbb{Z}}$;

- right and left fronts of the propagation process ignited at (x_0, t_0) the processes $(i_t^{0,+})_{t \geq 0}$ and $(i_t^{0,-})_{t \geq 0}$, where for $t \geq 0$

$$i_t^{0,+} = \max \left\{ i \geq 0 : \check{\zeta}_t^{\lambda,\pi,0}(i) = 2 \right\},$$

$$i_t^{0,-} = \min \left\{ i \leq 0 : \check{\zeta}_t^{\lambda,\pi,0}(i) = 2 \right\}.$$

The processes $(i_t^{0,+})_{t \geq 0}$ and $(-i_t^{0,-})_{t \geq 0}$ are Poisson processes with parameter π ;

- burning times of the propagation process ignited at (x_0, t_0) the sequence $(T_i^0)_{i \in \mathbb{Z}}$ where, for $i \in \mathbb{Z}$,

$$T_i^0 = \inf \left\{ s \geq 0 : \check{\zeta}_s^{\lambda,\pi,0}(i) = 2 \right\}$$

$$= \begin{cases} \inf \left\{ s \geq 0 : i_s^{0,+} = i \right\} & \text{if } i \geq 0, \\ \inf \left\{ s \geq 0 : i_s^{0,-} = i \right\} & \text{if } i \leq 0. \end{cases}$$

Observe that $(T_i^0)_{i \in \mathbb{Z}}, (i_t^{0,+})_{t \geq 0}$ and $(-i_t^{0,-})_{t \geq 0}$ only depend on the propagation processes family $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$.

We then reformulate Lemmas 4.2, 4.3, 4.4 and 4.5 with the previous definition.

Definition 4.7. Consider two families of Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all these processes being independent. Let $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$ and $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$ be the propagation process ignited at (x_0, t_0) , recall Definition 4.6.

- We define, for $T > 0$, $\Omega_{\lambda,\pi}^{P,T}(x_0, t_0) := \Omega_{\lambda,\pi}^{P,T}$, where $\Omega_{\lambda,\pi}^{P,T}$ is defined as in Lemma 4.2, using the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Lemma 4.2 implies that for all $\delta > 0$, $\mathbb{P} \left[\Omega_{\lambda,\pi}^{P,T}(x_0, t_0) \right] \geq 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

- We define, for $A, B > 0$, $\Omega_{\lambda,\pi}^{P,A,B}(x_0, t_0) := \Omega_{\lambda,\pi}^{P,A,B}$, where $\Omega_{\lambda,\pi}^{P,A,B}$ is defined as in Lemma 4.3, using the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Lemma 4.3 implies that for all $\delta > 0$, $\mathbb{P} \left[\Omega_{\lambda,\pi}^{P,A,B}(x_0, t_0) \right] \geq 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

- We define, for $z_0 \in [0, 1]$ and $\gamma \in (0, 1)$, $\Omega_{\lambda,\pi}^{P,T,\gamma}(x_0, t_0) := \Omega_{\lambda,\pi}^{P,T,\gamma}$, where $\Omega_{\lambda,\pi}^{P,T,\gamma}$ is defined as in Lemma 4.4, using the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Lemma 4.4 implies that for all $\delta > 0$, $\mathbb{P} \left[\Omega_{\lambda,\pi}^{P,T,\gamma}(x_0, t_0) \right] \geq 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

- We define, for $z_0 \in (0, 1]$ and $z \in (0, z_0)$, $\Omega_{\lambda,\pi}^{P,z}(x_0, t_0) := \Omega_{\lambda,\pi}^{P,z}$, where $\Omega_{\lambda,\pi}^{P,z}$ is defined as in Lemma 4.5, using the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Lemma 4.5 implies that for all $\delta > 0$, $\mathbb{P} \left[\Omega_{\lambda,\pi}^{P,z}(x_0, t_0) \right] \geq 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

Finally, we define the destroyed component by a fire starting on $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$. Indeed, knowing the sequence of burning times $(T_i)_{i \in \mathbb{Z}}$ and conditionally on a suitable event defined above, we can localize the set of sites which are burning by a fire.

Definition 4.8. Consider a family of independent Poisson processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with rate π . Let $(x_0, t_0) \in \mathbb{R} \times [0, T]$ and let $(T_i^0)_{i \in \mathbb{Z}}$ be the burning times of the propagation process ignited at (x_0, t_0) . For a \mathbb{N} -valued process $(\eta_t(i))_{t \geq 0, i \in \mathbb{Z}}$, we define

$$C^P((\eta_t(i))_{i \in \mathbb{Z}, t \geq 0}, (x_0, t_0)) = \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d \rrbracket \quad (4.12)$$

where

$$\begin{aligned} i^g &= \max \left\{ i \leq 0 : \eta_{\mathbf{a}_\lambda t_0 + T_i^0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = 0 \right\} + 1, \\ i^d &= \min \left\{ i \geq 0 : \eta_{\mathbf{a}_\lambda t_0 + T_i^0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = 0 \right\} - 1. \end{aligned}$$

We will use this definition with the (λ, π) -FFP: on a suitable event, $C^P((\eta_t^{\lambda, \pi}(i))_{i \in \mathbb{Z}, t \geq 0}, (x_0, t_0))$ is exactly the component destroyed by a match falling in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$, see the comments below.

Let now $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ be the (λ, π) -FFP. Let $(x_0, t_0) \in \mathbb{R} \times [0, \infty)$ be fixed in the rest of the section. Assume that a match falls in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at some time $\mathbf{a}_\lambda t_0$. Then, on an appropriate event and regardless of the other phenomena, fires propagate with the good speed while they spread in occupied zones. Indeed, consider $(\check{\zeta}_t^{\lambda, \pi, 0}(i))_{t \geq 0, i \in \mathbb{Z}}$ the propagation process ignited at (x_0, t_0) , the associated right front $(i_t^{0, +})_{t \geq 0}$ and left front $(i_t^{0, -})_{t \geq 0}$ and the associated burning times $(T_i^0)_{i \in \mathbb{Z}}$. Remark that $T_{i - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0$ is the time needed for the fire starting in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$ to reach i .

Microscopic fire: we describe here the effect of a microscopic fire in the discrete process in the different regimes. Let $\lambda \in (0, 1]$ and $\pi \geq 1$.

Micro(p): here we focus on the regime $\mathcal{R}(p)$, for some $p > 0$. Set $\kappa_{\lambda, \pi}^0 = \frac{\mathbf{m}_\lambda}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$. Assume that

- ▷ there are $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = \eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$ for all $t \in [t_0, t_0 + \kappa_{\lambda, \pi}^0]$,
- ▷ there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ at time $\mathbf{a}_\lambda t_0 -$,
- ▷ no other match falls in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^0)]$.

Then, on $\Omega_{\lambda, \pi}^{P, T}(x_0, t_0)$, we have

$$C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0)) \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket.$$

Furthermore, $\eta_{\mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^0)}^{\lambda, \pi}(i) \leq 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ and the fire destroys exactly the component $C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$.

Indeed, since $\mathbf{m}_\lambda = \mathbf{a}_\lambda \pi (\kappa_{\lambda, \pi}^0 - \varepsilon_\lambda)$, on $\Omega_{\lambda, \pi}^{P, T}(x_0, t_0)$, there holds that $T_{i_1}^0 \leq \mathbf{a}_\lambda \kappa_{\lambda, \pi}^0$ and $T_{i_2}^0 \leq \mathbf{a}_\lambda \kappa_{\lambda, \pi}^0$ (the left front satisfies $i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^0}^- \leq i_1$ and the right front satisfies $i_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^0}^+ \geq i_2$, thanks to Lemma 4.2). Consequently,

$$C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0)) := \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d \rrbracket \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$$

where i^g and i^d are defined in Definition 4.8. Observe now that, by construction, for all $i \in \llbracket i^g, i^d \rrbracket$

$$\eta_{\mathbf{a}_\lambda t_0 + T_i^0}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = 2 = \check{\zeta}_{T_i^0}^{\lambda, \pi, 0}(i)$$

and $\eta_{\mathbf{a}_\lambda t_0 + T_{i^g - 1}^0}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g - 1) = 0 = \eta_{\mathbf{a}_\lambda t_0 + T_{i^d + 1}^0}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d + 1)$. Recall that on $\Omega_{\lambda, \pi}^{T, P}(x_0, t_0)$, a spark at time $t \in [0, \mathbf{a}_\lambda T]$ for the process $(\check{\zeta}_t^{\lambda, \pi, 0}(i))_{t \geq 0, i \in \mathbb{Z}}$ has vacant neighbors. Since for all $i \in \llbracket i^g, i^d \rrbracket$, the processes $(\check{\zeta}_t^{\lambda, \pi, 0}(i))_{t \geq 0}$ and $(\eta_{\mathbf{a}_\lambda t_0 + t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i))_{t \geq 0}$ evolve with the same seed processes and the same propagation processes after burning for the first time until $\mathbf{a}_\lambda \kappa_{\lambda, \pi}^0$, a straightforward observation shows that for all $i \in \llbracket i^g + 1, i^d - 1 \rrbracket$,

$$\eta_{\mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^0)}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i) = \check{\zeta}_{\mathbf{a}_\lambda \kappa_{\lambda, \pi}^0}^{\lambda, \pi, 0}(i)$$

and a site $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket \setminus C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$ can't be burnt during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^0)]$. Observe also that i^g and i^d burn exactly once during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^0)]$

(because the site $i^d + 1$ is vacant at time $T_{i^d+1}^0$ and $i^g - 1$ is vacant at time $T_{i^g-1}^0$ with $T_{i^g}^0 \vee T_{i^d}^0 \leq \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$).

On $\Omega_{\lambda,\pi}^{P,T}(x_0, t_0)$, there is no more burning tree in $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket \supset \llbracket i^g, i^d \rrbracket$ at time $\mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$ for the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$ (because $\mathbf{m}_\lambda = \mathbf{a}_\lambda \pi(\kappa_{\lambda,\pi}^0 - \varepsilon_\lambda)$) and consequently, it also holds true in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d \rrbracket$ at time $\mathbf{a}_\lambda(t_0 + \kappa_{\lambda,\pi}^0)$ for the process $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$.

All this implies that, on $\Omega_{\lambda,\pi}^{P,T}(x_0, t_0)$, $\eta_{\mathbf{a}_\lambda(t_0 + \kappa_{\lambda,\pi}^0)}^{\lambda,\pi}(i) \leq 1$ for all $i \in C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$ and therefore for all $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$.

Micro(0): here we focus on the regime $\mathcal{R}(0)$. Let $A, B > 0$ and recall that, for $A > 0$, $\varkappa_{\lambda,\pi}^A = \frac{\mathbf{n}_\lambda A}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$. Assume that

- ▷ there are $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = \eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$ for all $t \in [t_0, t_0 + \varkappa_{\lambda,\pi}^{A \vee B}]$,
- ▷ there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ at time $\mathbf{a}_\lambda t_0 -$,
- ▷ no other match falls in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \varkappa_{\lambda,\pi}^{A \vee B})]$.

Then, on $\Omega_{\lambda,\pi}^{P,A,B}(x_0, t_0)$, we have

$$C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0)) \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket.$$

Furthermore, $\eta_{\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda,\pi}^{A \vee B})}^{\lambda,\pi}(i) \leq 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ and the fire destroys exactly the zone $C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$.

Indeed, this can be checked exactly as above (replace $\kappa_{\lambda,\pi}^0$ by $\varkappa_{\lambda,\pi}^{A \vee B}$ and $\Omega_{\lambda,\pi}^{P,T}(x_0, t_0)$ by $\Omega_{\lambda,\pi}^{P,A,B}(x_0, t_0)$).

Micro(∞, z_0): here we focus on the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in (0, 1]$ (in the case $z_0 = 0$, there are only fires of the second kind). Let $0 < z < z_0$ and recall that $\kappa_{\lambda,\pi}^z = \frac{1}{\lambda^z \mathbf{a}_\lambda \pi} + \varepsilon_\lambda$. Assume that

- ▷ there are $-\lfloor \lambda^{-z} \rfloor < i_1 < 0 < i_2 < \lfloor \lambda^{-z} \rfloor$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = \eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$ for all $t \in [t_0, t_0 + \kappa_{\lambda,\pi}^z]$,
- ▷ there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ at time $\mathbf{a}_\lambda t_0 -$,
- ▷ no other match falls in $\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \kappa_{\lambda,\pi}^z)]$.

Then, on $\Omega_{\lambda,\pi}^{P,z}(x_0, t_0)$, as above (replace $\kappa_{\lambda,\pi}^0$ by $\kappa_{\lambda,\pi}^z$ and $\Omega_{\lambda,\pi}^{P,T}(x_0, t_0)$ by $\Omega_{\lambda,\pi}^{P,z}(x_0, t_0)$)

$$C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0)) := \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^g, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i^d \rrbracket \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket.$$

Furthermore, $\eta_{\mathbf{a}_\lambda(t_0 + \kappa_{\lambda,\pi}^z)}^{\lambda,\pi}(i) \leq 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ and the fire destroys exactly the zone $C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$.

More precisely, on $\Omega_{\lambda,\pi}^{P,z}(x_0, t_0)$, for the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$, all site $i \in \llbracket i_{\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z}^{0,-} + 1, i_{\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z}^{0,+} - 1 \rrbracket$ burns exactly once before $\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z$. Thus, for the process $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0, i \in \mathbb{Z}}$, there is no spark in $\llbracket i_{\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z}^{0,-} + 1, i_{\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z}^{0,+} - 1 \rrbracket$ at any time $t \in [0, \mathbf{a}_\lambda \kappa_{\lambda,\pi}^z]$.

Since, for all $i \in \llbracket i^g, i^d \rrbracket$, the processes $(\check{\zeta}_t^{\lambda,\pi,0}(i))_{t \geq 0}$ and $(\eta_{\mathbf{a}_\lambda t_0 + t}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i))_{t \geq 0}$ evolve with the same seed processes and the same propagation processes after burning for the first time until $\mathbf{a}_\lambda \kappa_{\lambda,\pi}^z$, a straightforward observation shows that, for all $t \in [\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \kappa_{\lambda,\pi}^z)]$, and all $i \in C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$, for $i \geq \lfloor \mathbf{n}_\lambda x_0 \rfloor$,

$$\eta_t^{\lambda,\pi}(i) = \begin{cases} \min(\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) + N_{t+\mathbf{a}_\lambda t_0-}^S(i) - N_{\mathbf{a}_\lambda t_0}^S(i), 1) & \text{if } \mathbf{a}_\lambda t_0 \leq t < \mathbf{a}_\lambda t_0 + T_{i-\lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \\ 2 & \text{if } \mathbf{a}_\lambda t_0 + T_{i-\lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \leq t < \mathbf{a}_\lambda t_0 + T_{i+1-\lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \\ \min(N_t^S(i) - N_{T_{i+1-\lfloor \mathbf{n}_\lambda x_0 \rfloor}^0}^S(i), 1) & \text{if } \mathbf{a}_\lambda t_0 + T_{i+1-\lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \leq t \leq \mathbf{a}_\lambda(t_0 + \kappa_{\lambda,\pi}^z), \end{cases}$$

and, for $i \leq \lfloor \mathbf{n}_\lambda x_0 \rfloor$,

$$\eta_t^{\lambda, \pi}(i) = \begin{cases} \min(\eta_{\mathbf{a}_\lambda t_0 -}^{\lambda, \pi}(i) + N_{t + \mathbf{a}_\lambda t_0}^S(i) - N_{\mathbf{a}_\lambda t_0}^S(i), 1) & \text{if } \mathbf{a}_\lambda t_0 \leq t < \mathbf{a}_\lambda t_0 + T_{i - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \\ 2 & \text{if } \mathbf{a}_\lambda t_0 + T_{i - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \leq t < \mathbf{a}_\lambda t_0 + T_{i - 1 - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \\ \min(N_t^S(i) - N_{T_{i - 1 - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0}^S(i), 1) & \text{if } \mathbf{a}_\lambda t_0 + T_{i - 1 - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0 \leq t \leq \mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^z), \end{cases}$$

Finally, for $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket \setminus C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$ and $t \in [\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \kappa_{\lambda, \pi}^z)]$, $\eta_t^{\lambda, \pi}(i)$ is nothing but $\min(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) + N_{t + \mathbf{a}_\lambda t_0}^S(i) - N_{\mathbf{a}_\lambda t_0}^S(i), 1)$.

Macroscopic fire: let $\lambda \in (0, 1]$ and $\pi \geq 1$. Recall that, for $x > x_0$, $T_{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0$ is the time needed for the fire starting in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$ to reach $\lfloor \mathbf{n}_\lambda x \rfloor$.

Macro(p): here we focus on the regime $\mathcal{R}(p)$, for some $p > 0$. On $\Omega_{\lambda, \pi}^{P, T}(x_0, t_0)$, if $0 \leq x - x_0 \leq (T - t_0 - \varepsilon_\lambda) \frac{\mathbf{a}_\lambda \pi}{\mathbf{n}_\lambda}$, there holds that

$$\frac{\mathbf{a}_\lambda t_0 + T_{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0}{\mathbf{a}_\lambda} \in [t_0 + \frac{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, t_0 + \frac{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$$

and observe that, when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$,

$$[t_0 + \frac{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, t_0 + \frac{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda] \simeq \{t_0 + p(x - x_0)\}.$$

This is just a rewriting of Lemma 4.2.

Macro(0): here we focus on the regime $\mathcal{R}(0)$. On $\Omega_{\lambda, \pi}^{P, A, B}(x_0, t_0)$, for some $B > x - x_0$ and $A > 0$, there holds that

$$\frac{\mathbf{a}_\lambda t_0 + T_{\lfloor \mathbf{n}_\lambda x \rfloor - \lfloor \mathbf{n}_\lambda x_0 \rfloor}^0}{\mathbf{a}_\lambda} \in [t_0, t_0 + \varkappa_{\lambda, \pi}^B]$$

and observe that $[t_0, t_0 + \varkappa_{\lambda, \pi}^B] \simeq \{t_0\}$ when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

Besides, assume that

- ▷ there are $\lfloor \mathbf{n}_\lambda(x_0 - A) \rfloor < i_1 < \lfloor \mathbf{n}_\lambda x_0 \rfloor < i_2 < \lfloor \mathbf{n}_\lambda(x_0 + B) \rfloor$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(i_1) = \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(i_2) = 0$ for all $s \in [t_0, t_0 + \varkappa_{\lambda, \pi}^{A \vee B}]$,
- ▷ there is no burning tree in $\llbracket i_1, i_2 \rrbracket$ at time $\mathbf{a}_\lambda t_0$,
- ▷ no other match falls in $\llbracket i_1, i_2 \rrbracket$ during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \varkappa_{\lambda, \pi}^{A \vee B})]$.

Then, on $\Omega_{\lambda, \pi}^{P, A, B}(x_0, t_0)$, we have

$$C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0)) \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket.$$

Furthermore, $\eta_{\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda, \pi}^{A \vee B})}^{\lambda, \pi}(i) \leq 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket$ and the fire destroys exactly the zone $C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_0, t_0))$.

This can be shown exactly as in the case **Micro(p)** (the two statement are very similar).

Macro(∞, z_0): here we focus on fires of second kind in the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in [0, 1]$.

Let $\gamma \in (0, 1)$, on $\Omega_{\lambda, \pi}^{P, T, \gamma}(x_0, t_0)$, there holds that

$$x_0 - \frac{\mathbf{m}_\lambda^\gamma}{\mathbf{n}_\lambda} \leq \frac{\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_{\mathbf{a}_\lambda T}^{0, -}}{\mathbf{n}_\lambda} \leq x_0 \leq \frac{\lfloor \mathbf{n}_\lambda x_0 \rfloor + 1 + i_{\mathbf{a}_\lambda T}^{0, +}}{\mathbf{n}_\lambda} \leq x_0 + \frac{\mathbf{m}_\lambda^\gamma}{\mathbf{n}_\lambda}$$

and observe that $\mathbf{m}_\lambda^\gamma / \mathbf{n}_\lambda \leq \gamma$: this is just a rewriting of Lemma 4.4. Thus, since γ can be chosen arbitrarily small, in the regime $\mathcal{R}(\infty, z_0)$, fires have only a local effect.

5 Localization of the (λ, π) -FFP

Recall that $\mathbf{a}_\lambda, \mathbf{n}_\lambda$ and \mathbf{m}_λ are defined in (2.1), (2.2) and (2.5). For $A > 0$, we set $A_\lambda = \lfloor \mathbf{A}\mathbf{n}_\lambda \rfloor$ and $I_A^\lambda = \llbracket -A_\lambda, A_\lambda \rrbracket$. For $i \in \mathbb{Z}$, we set $i_\lambda = \lfloor i/\mathbf{n}_\lambda, (i+1)/\mathbf{n}_\lambda \rfloor$ and $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$.

We first introduce the (λ, π, A) -FFP.

Definition 5.1. Let $\lambda \in (0, 1], \pi \geq 1$ and $A > 0$ be fixed. For each $i \in I_A^\lambda$, we consider three independent Poisson processes, $N^S(i) = (N_t^S(i))_{t \geq 0}$, $N^M(i) = (N_t^M(i))_{t \geq 0}$ and $N^P(i) = (N_t^P(i))_{t \geq 0}$ of respective parameters $1, \lambda$ and π , all these processes being independent. Consider a $\{0, 1, 2\}$ -valued process $(\eta_t^{\lambda, \pi, A}(i))_{t \geq 0, i \in I_A^\lambda}$ such that a.s., for all $i \in I_A^\lambda$, $(\eta_t^{\lambda, \pi, A}(i))_{t \geq 0}$ is càdlàg. We say that $(\eta_t^{\lambda, \pi, A}(i))_{t \geq 0, i \in I_A^\lambda}$ is a (λ, π, A) -FFP if a.s., for all $i \in I_A^\lambda$, all $t \geq 0$

$$\begin{aligned} \eta_t^{\lambda, \pi, A}(i) = & \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi, A}(i)=0\}} dN_s^S(i) + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi, A}(i)=1\}} dN_s^M(i) \\ & + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi, A}(i+1)=2, \eta_{s-}^{\lambda, \pi, A}(i)=1\}} dN_s^P(i+1) + \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi, A}(i-1)=2, \eta_{s-}^{\lambda, \pi, A}(i)=1\}} dN_s^P(i-1) \\ & - 2 \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda, \pi, A}(i)=2\}} dN_s^P(i) \end{aligned}$$

with the convention $N_t^S(A_\lambda + 1) = N_t^S(-A_\lambda - 1) = 0$ for all $t \geq 0$.

For $\eta \in \{0, 1, 2\}^{I_A^\lambda}$ and $i \in I_A^\lambda$, we define the occupied connected component around i as

$$C_A(\eta, i) = \begin{cases} \emptyset & \text{if } \eta(i) = 0 \text{ or } 2, \\ \llbracket l_A(\eta, i), r_A(\eta, i) \rrbracket & \text{if } \eta(i) = 1, \end{cases}$$

where

$$\begin{aligned} l_A(\eta, i) &= (-A_\lambda) \vee (\sup\{k < i : \eta(k) = 0 \text{ or } 2\} + 1), \\ r_A(\eta, i) &= A_\lambda \wedge (\inf\{k > i : \eta(k) = 0 \text{ or } 2\} - 1). \end{aligned}$$

For $x \in [-A, A]$ and $t \geq 0$, we also introduce

$$D_t^{\lambda, \pi, A}(x) = \frac{1}{\mathbf{n}_\lambda} C_A \left(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}, \lfloor \mathbf{n}_\lambda x \rfloor \right) \subset [-A_\lambda/\mathbf{n}_\lambda, A_\lambda/\mathbf{n}_\lambda] \simeq [-A, A], \quad (5.1)$$

$$K_t^{\lambda, \pi, A}(x) = \frac{\left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda \rrbracket \cap I_A^\lambda : \eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(i) = 1 \right\} \right|}{\left| \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda \rrbracket \cap I_A^\lambda \right|} \in [0, 1], \quad (5.2)$$

$$Z_t^{\lambda, \pi, A}(x) = \frac{-\log(1 - K_t^{\lambda, \pi, A}(x))}{\log(1/\lambda)} \wedge 1 \in [0, 1]. \quad (5.3)$$

We now give a discrete version of Proposition 3.5. Recall Definition 4.1.

Proposition 5.2. Let $T > 0, \lambda \in (0, 1]$ and $\pi \geq 1$. For each $i \in \mathbb{Z}$, we consider three Poisson processes $N^S(i) = (N_t^S(i))_{t \geq 0}$, $N^M(i) = (N_t^M(i))_{t \geq 0}$ and $N^P(i) = (N_t^P(i))_{t \geq 0}$, all these processes being independent. Let $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ be the corresponding (λ, π) -FFP and for each $A > 0$, let $(\eta_t^{\lambda, \pi, A}(i))_{t \geq 0, i \in I_A^\lambda}$ be the corresponding (λ, π, A) -FFP. There are some constants $\alpha_T > 0$ and $C_T > 0$ such that for all $A \geq 1$, all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$, for some $p \geq 0$ (or to the regime $\mathcal{R}(\infty, z_0)$, for some $z_0 \in [0, 1]$),

$$\begin{aligned} \mathbb{P} \left[(\eta_t^{\lambda, \pi}(i))_{t \in [0, \mathbf{a}_\lambda T], i \in I_{A/2}^\lambda} = (\eta_t^{\lambda, \pi, A}(i))_{t \in [0, \mathbf{a}_\lambda T], i \in I_{A/2}^\lambda}, \right. \\ \left. (Z_t^{\lambda, \pi}(x), D_t^{\lambda, \pi}(x))_{t \in [0, T], x \in [-A/2, A/2]} = (Z_t^{\lambda, \pi, A}(x), D_t^{\lambda, \pi, A}(x))_{t \in [0, T], x \in [-A/2, A/2]} \right] \\ \geq 1 - C_T e^{-\alpha_T A}. \end{aligned}$$

Observe that the Proposition 5.2 holds for the three regimes, with the same scales but for different reasons. We thus distinguish the three regimes. The proof given for $p = 0$ can be adapted in order to work for $p > 0$, as in Proposition 3.5, but the proof given here for $p > 0$ is much simpler.

Proof in the regime $\mathcal{R}(p)$ for some $p > 0$. Consider the true (λ, π) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$. It of course suffices to prove the result for A large enough. Temporarily assume that for $a \in \mathbb{R}$, there is an event $\Omega_{a, T}^{\lambda, \pi}$, depending only on the Poisson processes $N_t^S(i)$, $N_t^M(i)$ and $N_t^P(i)$ for $t \in [0, \mathbf{a}_\lambda(T+2)]$ and

$$i \in \bar{J}_a^\lambda := \llbracket (a-1-2\frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a+1+2\frac{T-1}{p})\mathbf{n}_\lambda \rrbracket - 1],$$

such that

- (i) on $\Omega_{a, T}^{\lambda, \pi}$, a.s., there are $\iota^+ : [0, \mathbf{a}_\lambda T] \mapsto \bar{J}_a^\lambda$ non decreasing and $\iota^- : [0, \mathbf{a}_\lambda T] \mapsto \bar{J}_a^\lambda$ non increasing such that $\eta_t^{\lambda, \pi}(\iota_t^+) = 0$ or 2 and $\eta_t^{\lambda, \pi}(\iota_t^-) = 0$ or 2 for all $t \in [0, \mathbf{a}_\lambda T]$,
- (ii) there exists $q_T > 0$ such that for all $a \in \mathbb{R}$, we have $\mathbb{P}[\Omega_{a, T}^{\lambda, \pi}] \geq q_T$, for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

The proof is then concluded using similar argument as Step 3 in the proof of Proposition 3.5: thanks to point (ii), the probability that there are $-A+1+2\frac{T-1}{p} < a_1 < -A/2-1-2\frac{T-1}{p}$ and $A/2+1+2\frac{T-1}{p} < a_2 < A-1-2\frac{T-1}{p}$ with $\Omega_{a_1, T}^{\lambda, \pi}$ and $\Omega_{a_2, T}^{\lambda, \pi}$ realized is easily bounded from below by $1 - C_T e^{-\alpha T A}$. Next, on this event, a fire starting at the left of $\llbracket (a_1-1-2\frac{T-1}{p})\mathbf{n}_\lambda \rrbracket$ will never cross $\llbracket (a_1+1+2\frac{T-1}{p})\mathbf{n}_\lambda \rrbracket \leq \llbracket A\mathbf{n}_\lambda/2 \rrbracket$ (thanks to ι^+). Same thing holds on the right: a fire starting at the right of $\llbracket (a_2+1+2\frac{T-1}{p})\mathbf{n}_\lambda \rrbracket$ will never cross $\llbracket (a_2-1-2\frac{T-1}{p})\mathbf{n}_\lambda \rrbracket \geq \llbracket A\mathbf{n}_\lambda/2 \rrbracket$ (thanks to ι^-). Finally, the clusters $D_t^{\lambda, \pi}(x)$ and $D_t^{\lambda, \pi, A}(x)$ clearly coincide for all $x \in [-\frac{A}{2}, \frac{A}{2}]$ and all $t \in [0, T]$.

Step 1. Fix some $\alpha > 0$ small enough, say $\alpha = 0.001$. Define $\kappa_{\lambda, \pi}^0 = \mathbf{m}_\lambda/(\mathbf{a}_\lambda \pi) + \varepsilon_\lambda$ and assume that $\kappa_{\lambda, \pi}^0 \leq \alpha/2$.

For $\lambda > 0, \pi \geq 1$ and $a \in \mathbb{R}$, we define the event $\tilde{\Omega}_{a, T}^{\lambda, \pi}$ on which points 1 and 2 below are satisfied:

1. The family of Poisson processes $(N_t^M(i))_{t \in [0, \mathbf{a}_\lambda T], i \in \bar{J}_a^\lambda}$ has exactly 4 marks in \bar{J}_a^λ , and we call them $\{(X_1^\lambda, T_1^\lambda), (X_2^\lambda, T_2^\lambda), (X_3^\lambda, T_3^\lambda), (X_4^\lambda, T_4^\lambda)\}$, in such a way the match $(X_1^\lambda, T_1^\lambda)$ (resp. $(X_2^\lambda, T_2^\lambda)$) belongs to

$$\begin{aligned} & \llbracket (a - \frac{5}{6} - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a - \frac{2}{3} - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket \rrbracket \times [\mathbf{a}_\lambda(\frac{3}{4} + \alpha), \mathbf{a}_\lambda(1 - \alpha)] \\ \text{(resp. } & \llbracket (a + \frac{2}{3} + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a + \frac{5}{6} + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket \rrbracket \times [\mathbf{a}_\lambda(\frac{3}{4} + \alpha), \mathbf{a}_\lambda(1 - \alpha)]), \end{aligned}$$

and the match $(X_3^\lambda, T_3^\lambda)$ (resp. $(X_4^\lambda, T_4^\lambda)$) belongs to

$$\begin{aligned} & \llbracket (a - \frac{1}{2} - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket + 1, \llbracket (a - \frac{1}{3} - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket \rrbracket \times [\mathbf{a}_\lambda(1 + \alpha), \mathbf{a}_\lambda(\frac{3}{2} - \alpha)] \\ \text{(resp. } & \llbracket (a + \frac{1}{3} + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a + \frac{1}{2} + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket - 1 \rrbracket \times [\mathbf{a}_\lambda(1 + \alpha), \mathbf{a}_\lambda(\frac{3}{2} - \alpha)]). \end{aligned}$$

2. The family of Poisson processes $(N_t^S(i))_{t \geq 0, i \in \bar{J}_a^\lambda}$ satisfies

- (a) for $k = 1, 2$, for all $i \in \llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket$, $N_{T_k^\lambda}^S(i) > 0$;
- (b) for $k = 1, 2$, there are $i_1^k \in \llbracket X_k^\lambda - \mathbf{m}_\lambda + 1, X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor - 1 \rrbracket$ and $i_2^k \in \llbracket X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor + 1, X_k^\lambda + \mathbf{m}_\lambda - 1 \rrbracket$ such that $N_{T_k^\lambda + \mathbf{a}_\lambda \kappa_{\lambda, \pi}^0}^S(i_1^k) = N_{T_k^\lambda + \mathbf{a}_\lambda \kappa_{\lambda, \pi}^0}^S(i_2^k) = 0$;
- (c) for $k = 1, 2$, there is $i_3^k \in \llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket$ such that $N_{3\mathbf{a}_\lambda/2}^S(i_3^k) - N_{T_k^\lambda}^S(i_3^k) = 0$;

(d) for all $i \in \llbracket (a-1 - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a+1 + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, N_{\mathbf{a}_\lambda(1+\alpha)}^S(i) > 0$.

We now introduce the event $\Omega_{a,T}^P(\lambda, \pi)$ on which all these four fires propagate at the good speed

$$\Omega_{a,T}^P(\lambda, \pi) = \bigcap_{i=1}^4 \Omega_{\lambda,\pi}^{P,T} \left(\frac{X_i^\lambda}{\mathbf{n}_\lambda}, \frac{T_i^\lambda}{\mathbf{a}_\lambda} \right)$$

recall Definition 4.7. We finally set

$$\Omega_{a,T}^{\lambda,\pi} = \tilde{\Omega}_{a,T}^{\lambda,\pi} \cap \Omega_{a,T}^P(\lambda, \pi).$$

Step 2. We now prove that on $\Omega_{a,T}^{\lambda,\pi}$, there exist $(\iota_t^+)_{t \in [0, \mathbf{a}_\lambda T]}$ and $(\iota_t^-)_{t \in [0, \mathbf{a}_\lambda T]}$ which satisfy (i).

Indeed, sites i_1^1 and i_2^1 are vacant until $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$ because we start from an vacant initial configuration and 2-(b). On the one hand, they protect the zone $\llbracket i_1^1 + 1, i_2^1 - 1 \rrbracket$ and thus, the zone $\llbracket X_1^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_1^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket \subset \llbracket i_1^1 + 1, i_2^1 - 1 \rrbracket$ is completely filled at time T_1^λ , thanks to 2-(a). On the other hand, on $\Omega_{\lambda,\pi}^{P,T}(X_1^\lambda/\mathbf{n}_\lambda, T_1^\lambda/\mathbf{a}_\lambda)$, as seen in **Micro**(p) in Subsection 4.4,

- ▷ the match falling on X_1^λ at time T_1^λ destroys entirely the zone $\llbracket X_1^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_1^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket$ before $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$ (it is still protected by i_1^1 and i_2^1),
- ▷ the fire does not affect the zone outside $\llbracket i_1^1, i_2^1 \rrbracket$,
- ▷ there is no more burning tree in the zone $\llbracket i_1^1, i_2^1 \rrbracket$ at time $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$.

Then, since no seed fall on i_3^1 during $[T_1^\lambda, 3\mathbf{a}_\lambda/2]$, i_3^1 remains vacant since it burnt (this happened between T_1^λ and $T_1^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$) until time $3\mathbf{a}_\lambda/2$, thanks to 2-(c).

Remark that same considerations holds around X_2^λ : the match falling in X_2^λ at time T_2^λ doesn't affect the zone outside $\llbracket i_1^2, i_2^2 \rrbracket$ (because they remain vacant until time $T_2^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0$), and i_3^2 remains vacant during $[T_2^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^0, 3\mathbf{a}_\lambda/2]$.

All this implies that the zone $\llbracket (a - \frac{1}{2} - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a + \frac{1}{2} + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket$ is protected from all the fire until $3\mathbf{a}_\lambda/2$ (except possibly those falling at $(X_3^\lambda, T_3^\lambda)$ and $(X_4^\lambda, T_4^\lambda)$). Thus, thanks to 2-(d), the zone $\llbracket (a - \frac{1}{2} - \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket, \llbracket (a + \frac{1}{2} + \frac{T-1}{p})\mathbf{n}_\lambda \rrbracket$ is completely occupied at time $\mathbf{a}_\lambda(1 + \alpha)$.

Since now, on $\Omega_{\lambda,\pi}^{P,T} \left(\frac{X_3^\lambda}{\mathbf{n}_\lambda}, \frac{T_3^\lambda}{\mathbf{a}_\lambda} \right)$, the right front $(i_t^{3,+})_{t \geq 0}$ of the fire ignited at $(X_3^\lambda/\mathbf{n}_\lambda, T_3^\lambda/\mathbf{a}_\lambda)$ satisfies

$$i_{\mathbf{a}_\lambda T - T_3^\lambda}^{3,+} \leq \pi(\mathbf{a}_\lambda T - T_3^\lambda + \mathbf{a}_\lambda \varepsilon_\lambda) \leq \mathbf{a}_\lambda \pi(T - 1 - \alpha + \varepsilon_\lambda),$$

recall Lemma 4.2, then $i_{\mathbf{a}_\lambda T - T_3^\lambda}^{3,+} \leq (T-1) \frac{\mathbf{n}_\lambda}{p}$ as soon as $\left| \frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p \right| \leq p \frac{\alpha}{2(T-1)}$ (recall that $2\varepsilon < \alpha$). This in particular implies that

$$X_3^\lambda + i_{\mathbf{a}_\lambda T - T_3^\lambda}^{3,+} \leq \lfloor (a - \frac{1}{3} - \frac{T-1}{p})\mathbf{n}_\lambda \rfloor + (T-1) \frac{\mathbf{n}_\lambda}{p} < \lfloor \mathbf{n}_\lambda a \rfloor.$$

Similarly, on $\Omega_{\lambda,\pi}^{P,T} \left(\frac{X_4^\lambda}{\mathbf{n}_\lambda}, \frac{T_4^\lambda}{\mathbf{a}_\lambda} \right)$ and for $\left| \frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p \right| \leq p \frac{\alpha}{2(T-1)}$, we clearly have

$$\lfloor \mathbf{n}_\lambda a \rfloor < \lfloor (a + \frac{1}{3} + \frac{T-1}{p})\mathbf{n}_\lambda \rfloor - (T-1) \frac{\mathbf{n}_\lambda}{p} \leq X_4^\lambda + i_{\mathbf{a}_\lambda T - T_4^\lambda}^{4,-}.$$

We easily deduce that for all $t \in [0, \mathbf{a}_\lambda T - T_3^\lambda]$, $\eta_{t+T_3^\lambda}^{\lambda,\pi}(X_3^\lambda + i_t^{3,+}) = 2$ and for all $t \in [0, \mathbf{a}_\lambda T - T_4^\lambda]$, $\eta_{t+T_4^\lambda}^{\lambda,\pi}(X_4^\lambda + i_t^{4,-}) = 2$.

Finally, we set, for all $t \in [0, \mathbf{a}_\lambda T]$

$$\iota_t^+ = \begin{cases} i_1^1 & \text{if } 0 \leq t < T_1^\lambda + \kappa_{\lambda,\pi}^0, \\ i_3^1 & \text{if } T_1^\lambda + \kappa_{\lambda,\pi}^0 \leq t < T_3^\lambda, \\ X_3^\lambda + i_{t-T_3^\lambda}^{3,+} & \text{if } T_3^\lambda \leq t \leq \mathbf{a}_\lambda T. \end{cases}$$

Clearly, $(\iota_t^+)_{t \in [0, \mathbf{a}_\lambda T]}$ is non decreasing, $\eta_s^{\lambda, \pi}(\iota_s^+)$ is 0 until T_3^λ and 2 between T_3^λ and $\mathbf{a}_\lambda T$. Similarly, we can choose

$$\iota_t^- = \begin{cases} i_2^2 & \text{if } 0 \leq t < T_2^\lambda + \kappa_{\lambda, \pi}^0, \\ i_3^2 & \text{if } T_2^\lambda + \kappa_{\lambda, \pi}^0 \leq t < T_4^\lambda, \\ X_4^\lambda + i_{t-T_4^\lambda}^{4, -} & \text{if } T_4^\lambda \leq t \leq \mathbf{a}_\lambda T. \end{cases}$$

Clearly, $(\iota_t^-)_{t \in [0, \mathbf{a}_\lambda T]}$ is non increasing, $\eta_s^{\lambda, \pi}(\iota_s^-)$ is 0 until T_4^λ and 2 between T_4^λ and $\mathbf{a}_\lambda T$.

Step 3. We now prove (ii). The quantity $\mathbb{P}[\Omega_{a, T}^{\lambda, \pi}]$ does obviously not depend on $a \in \mathbb{R}$ by spatial invariance. Then, we observe that we can construct N^M by using a Poisson measure π_M on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$, independent of N^S and N^P , by setting, for all $i \in \mathbb{Z}$,

$$N_t^M(i) = \pi_M(i_\lambda \times [0, t/\mathbf{a}_\lambda]).$$

Hence, the event on which N^M satisfies 1. contains the event $\Omega_{a, T}^M$ on which π_M has exactly 4 marks in $[a - 1 - 2\frac{T-1}{p}, a + 1 + 2\frac{T-1}{p}] \times [0, T]$, which can be called $(X_1, T_1), (X_2, T_2), (X_3, T_3)$ and (X_4, T_4) in such a way (X_1, T_1) (resp. (X_2, T_2)) belongs to

$$\begin{aligned} & [a - \frac{5}{6} - \frac{T-1}{p} + \alpha, a - \frac{2}{3} - \frac{T-1}{p} - \alpha] \times [\frac{3}{4} + \alpha, 1 - \alpha] \\ (\text{resp. } & [a + \frac{2}{3} + \frac{T-1}{p} + \alpha, a + \frac{5}{6} + \frac{T-1}{p} - \alpha] \times [\frac{3}{4} + \alpha, 1 - \alpha]), \end{aligned}$$

and (X_3, T_3) (resp. (X_4, T_4)) belongs to

$$\begin{aligned} & [a - \frac{1}{2} - \frac{T-1}{p} + \alpha, a - \frac{1}{3} - \frac{T-1}{p} - \alpha] \times [1 + \alpha, \frac{3}{2} - \alpha] \\ (\text{resp. } & [a + \frac{1}{3} + \frac{T-1}{p} + \alpha, a + \frac{1}{2} + \frac{T-1}{p} - \alpha] \times [1 + \alpha, \frac{3}{2} - \alpha]). \end{aligned}$$

Clearly, the probability $\mathbb{P}[\Omega_{a, T}^M]$ does not depend on a nor on λ and π and is positive. We then define $q_T > 0$ by

$$\mathbb{P}[\Omega_{a, T}^M] = 2q_T. \quad (\star)$$

We then use basic consideration on i.i.d. Poisson processes with rate 1 (we write \mathbb{P}_M for the conditional probability w.r.t. π_M) to show that point 2. occurs with high probability.

- For $k = 1, 2$, we have $T_k^\lambda \geq \mathbf{a}_\lambda(3/4 + \alpha)$ and

$$\mathbb{P}_M \left[\forall i \in \llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket, N_{T_k^\lambda}^S(i) > 0 \right] \geq (1 - \lambda^{3/4 + \alpha})^{2\lfloor \lambda^{-3/4} \rfloor + 1}$$

which tends to 1 when $\lambda \rightarrow 0$.

- For $k = 1, 2$, we have $T_k^\lambda + \mathbf{a}_\lambda \kappa_{\lambda, \pi}^0 \leq \mathbf{a}_\lambda(1 - \alpha/2)$ (recall that $\kappa_{\lambda, \pi}^0 \leq \alpha/2$) and

$$\begin{aligned} \mathbb{P}_M \left[\exists i_2^k \in \llbracket X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor + 1, X_k^\lambda + \mathbf{m}_\lambda - 1 \rrbracket, N_{T_k^\lambda + \mathbf{a}_\lambda \kappa_{\lambda, \pi}^0}^S(i_2^k) = 0 \right] \\ \geq 1 - (1 - \lambda^{1 - \alpha/2})^{\mathbf{m}_\lambda - \lfloor \lambda^{-3/4} \rfloor - 1} \end{aligned}$$

which tends to 1 when $\lambda \rightarrow 0$ (and similar computation for i_1^k).

- For $k = 1, 2$, we have $T_k^\lambda \geq \mathbf{a}_\lambda(3/4 + \alpha)$ and

$$\begin{aligned} \mathbb{P}_M \left[\exists i \in \llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket, N_{3\mathbf{a}_\lambda/2}^S(i) - N_{T_k^\lambda}^S(i) = 0 \right] \\ \geq 1 - (1 - \lambda^{3/4 - \alpha})^{2\lfloor \lambda^{-3/4} \rfloor + 1} \end{aligned}$$

which tends to 1 when $\lambda \rightarrow 0$;

- Finally,

$$\mathbb{P}_M \left[\forall i \in \llbracket \lfloor (a-1 - \frac{T-1}{p})\mathbf{n}_\lambda \rfloor, \lfloor (a+1 + \frac{T-1}{p})\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda(1+\alpha)}^S(i) > 0 \right] \\ = (1 - \lambda^{1+\alpha})^{(2+2\frac{T-1}{p})\mathbf{n}_\lambda}$$

which tends also to 1 when $\lambda \rightarrow 0$.

Next, since π_M is independent of the processes family $(N_t^S(i))_{i \in \mathbb{Z}, t \geq 0}$ and $(N_t^P(i))_{i \in \mathbb{Z}, t \geq 0}$, Lemma 4.2 directly imply that, for all $k = 1, \dots, 4$, $\mathbb{P}_M \left[\Omega_{\lambda, \pi}^{P, T}(X_k, T_k) \right]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

All this, together with (\star) , implies that $\mathbb{P} \left[\Omega_{a, T}^{\lambda, \pi} \right] \geq q_T > 0$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

In the end, for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$, the event $\Omega_{a, T}^{\lambda, \pi}$ depend only on the Poisson processes $N_t^S(i)$, $N_t^M(i)$ and $N_t^P(i)$ for $t \in [0, \mathbf{a}_\lambda(T+2)]$ and $i \in \bar{J}_a^\lambda$. This suffices to conclude the proof. \square

Proof in the regime $\mathcal{R}(\infty, z_0)$. Let $z_0 \in [0, 1]$. Consider the true (λ, π) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$. We introduce

$$J_a^\lambda = \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor (a+1)\mathbf{n}_\lambda \rfloor - 1 \rrbracket.$$

As above, for $a \in \mathbb{R}$, we are going to construct an event $\Omega_{a, T}^{\lambda, \pi}$ depending only on the Poisson processes $N_t^S(i)$, $N_t^M(i)$ and $N_t^P(i)$ for $t \in [0, \mathbf{a}_\lambda(T+2)]$ and $i \in J_a^\lambda$ such that

- (i) on $\Omega_{a, T}^{\lambda, \pi}$, there exists $\iota^+ : [0, \mathbf{a}_\lambda T] \mapsto J_a^\lambda$ non decreasing and $\iota^- : [0, \mathbf{a}_\lambda T] \mapsto J_a^\lambda$ non increasing such that $\eta_t^{\lambda, \pi}(\iota_t^+) = 0$ or 2 and $\eta_t^{\lambda, \pi}(\iota_t^-) = 0$ or 2 for all $t \in [0, \mathbf{a}_\lambda T]$,
- (ii) there exists $q_T > 0$ such that for all $a \in \mathbb{R}$, we have $\mathbb{P} \left[\Omega_{a, T}^{\lambda, \pi} \right] \geq q_T$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

The proof is then concluded as previously. We divide the proof in two cases.

Case 1: $z_0 \in [0, 1)$. We fix $\alpha = 0.001$ and $\gamma \in (0, \frac{1-z_0}{4})$. Recall that $\mathbf{m}_\lambda^\gamma = \lfloor \frac{\gamma}{\lambda^{\gamma+(1-\gamma)z_0}\mathbf{a}_\lambda} \rfloor \ll \mathbf{m}_\lambda$ and $\mathbf{m}_\lambda \ll \mathbf{n}_\lambda$.

Step 1. For $\lambda > 0, \pi \geq 1$ and $a \in \mathbb{R}$, we define the event $\tilde{\Omega}_{a, T}^{\lambda, \pi}$ on which points 1 and 2 below are satisfied:

1. The family of Poisson processes $(N_t^M(i))_{t \in [0, \mathbf{a}_\lambda T], i \in J_a^\lambda}$ has exactly 2 marks in J_a^λ , and we call them $(X_1^\lambda, T_1^\lambda), (X_2^\lambda, T_2^\lambda)$, in such a way that

$$(X_1^\lambda, T_1^\lambda) \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor + \mathbf{m}_\lambda, \lfloor (a + \frac{1}{2})\mathbf{n}_\lambda \rfloor - \mathbf{m}_\lambda - 1 \rrbracket \times [\mathbf{a}_\lambda(z_0 + 2\gamma), \mathbf{a}_\lambda(1 - \gamma)] \\ \text{and } (X_2^\lambda, T_2^\lambda) \in \llbracket \lfloor (a + \frac{1}{2})\mathbf{n}_\lambda \rfloor + \mathbf{m}_\lambda, \lfloor (a+1)\mathbf{n}_\lambda \rfloor - \mathbf{m}_\lambda - 1 \rrbracket \times [\mathbf{a}_\lambda(z_0 + 2\gamma), \mathbf{a}_\lambda(1 - \gamma)].$$

2. The family of Poisson processes $(N_t^S(i))_{t \geq 0, i \in J_a^\lambda}$ satisfies, for $k = 1, 2$,

- (a) for all $i \in \llbracket X_k^\lambda - \mathbf{m}_\lambda^\gamma, X_k^\lambda + \mathbf{m}_\lambda^\gamma \rrbracket, N_{T_k^\lambda}^S(i) > 0$;
- (b) there are $i_1^k \in \llbracket X_k^\lambda - \mathbf{m}_\lambda + 1, X_k^\lambda - \mathbf{m}_\lambda^\gamma - 1 \rrbracket$ and $i_2^k \in \llbracket X_k^\lambda + \mathbf{m}_\lambda^\gamma + 1, X_k^\lambda + \mathbf{m}_\lambda - 1 \rrbracket$ such that $N_{\mathbf{a}_\lambda(1-\gamma)}^S(i_1^k) = N_{\mathbf{a}_\lambda(1-\gamma)}^S(i_2^k) = 0$.

We now introduce the event on which all of these two fires propagate at the correct speed,

$$\Omega_{a, T}^P(\lambda, \pi) = \Omega_{\lambda, \pi}^{P, T, \gamma} \left(\frac{X_1^\lambda}{\mathbf{n}_\lambda}, \frac{T_1^\lambda}{\mathbf{a}_\lambda} \right) \cap \Omega_{\lambda, \pi}^{P, T, \gamma} \left(\frac{X_2^\lambda}{\mathbf{n}_\lambda}, \frac{T_2^\lambda}{\mathbf{a}_\lambda} \right).$$

We finally set

$$\Omega_{a,T}^{\lambda,\pi} = \tilde{\Omega}_{a,T}^{\lambda,\pi} \cap \Omega_{a,T}^P(\lambda, \pi).$$

Step 2. We now prove that on $\Omega_{a,T}^{\lambda,\pi}$, (i) holds.

For $k = 1, 2$, thanks to 2-(b), the sites i_1^k and i_2^k remain vacant until $\mathbf{a}_\lambda(1-\gamma) > T_k^\lambda$. Thus, no fire can affect the zone $\llbracket X_k^\lambda - \mathbf{m}_\lambda^\gamma, X_k^\lambda + \mathbf{m}_\lambda^\gamma \rrbracket$ during $[0, \mathbf{a}_\lambda(1-\gamma)]$. Hence, the zone $\llbracket X_k^\lambda - \mathbf{m}_\lambda^\gamma, X_k^\lambda + \mathbf{m}_\lambda^\gamma \rrbracket$ is completely filled at time $T_k^\lambda -$, thanks to 2-(a). On $\Omega_{\lambda,\pi}^{P,T,\gamma} \left(\frac{X_k^\lambda}{\mathbf{n}_\lambda}, \frac{T_k^\lambda}{\mathbf{a}_\lambda} \right) \subset \Omega_{a,T}^P(\lambda, \pi)$, the fire starting in X_k^λ at time T_k^λ does not affect the zone outside $\llbracket X_k^\lambda - \mathbf{m}_\lambda^\gamma, X_k^\lambda + \mathbf{m}_\lambda^\gamma \rrbracket$ during $[0, \mathbf{a}_\lambda T]$, recall **Macro**(∞, z_0) in Subsection 4.4. Since $X_2^\lambda - X_1^\lambda \geq 2\mathbf{m}_\lambda^\gamma \geq 2\mathbf{m}_\lambda^\gamma + 1$, we deduce that $\eta_s^{\lambda,\pi}(X_1^\lambda + i_{s-T_1^\lambda}^{1,+}) = 2$ for all $s \in [T_1^\lambda, \mathbf{a}_\lambda T]$ and $\eta_s^{\lambda,\pi}(X_2^\lambda + i_{s-T_2^\lambda}^{2,-}) = 2$ for all $s \in [T_2^\lambda, \mathbf{a}_\lambda T]$.

Finally, we set, for all $t \in [0, \mathbf{a}_\lambda T]$

$$\iota_t^+ = \begin{cases} i_1^1 & \text{if } 0 \leq t < T_1^\lambda, \\ X_1^\lambda + i_{t-T_1^\lambda}^{1,+} & \text{if } T_1^\lambda \leq t \leq \mathbf{a}_\lambda T. \end{cases}$$

The process $(\iota_t^+)_{t \in [0, \mathbf{a}_\lambda T]}$ is non decreasing, $\eta_s^{\lambda,\pi}(\iota_s^+)$ is 0 for $s \in [0, T_1^\lambda)$ and 2 for $s \in [T_1^\lambda, \mathbf{a}_\lambda T]$.

Similarly, we set for all $t \in [0, \mathbf{a}_\lambda T]$,

$$\iota_t^- = \begin{cases} i_2^2 & \text{if } 0 \leq t < T_2^\lambda, \\ X_2^\lambda + i_{t-T_2^\lambda}^{2,-} & \text{if } T_2^\lambda \leq t \leq \mathbf{a}_\lambda T, \end{cases}$$

which also satisfies the requirements.

Step 3. The event $\Omega_{a,T}^{\lambda,\pi}$ also satisfies point (ii).

Indeed, the quantity $\mathbb{P}[\Omega_{a,T}^{\lambda,\pi}]$ does obviously not depend on $a \in \mathbb{R}$ by spatial invariance. As previously, we can construct N^M by using a Poisson measure π_M on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$, independent of N^S and N^P , by setting, for all $i \in \mathbb{Z}$,

$$N_t^M(i) = \pi_M(i_\lambda \times [0, t/\mathbf{a}_\lambda]).$$

Hence, the event on which N^M satisfies 1. contains the event $\Omega_{a,T}^M$ on which π_M has exactly 2 marks in $[a, a+1] \times [0, T]$, which can be called (X_1, T_1) and (X_2, T_2) such that (remark that $\gamma < 1/4$)

$$(X_1, T_1) \in [a + \gamma, a + \frac{1}{2} - \gamma] \times [z_0 + 2\gamma, 1 - \gamma]$$

$$\text{and } (X_2, T_2) \in [a + \frac{1}{2} + \gamma, a + 1 - \gamma] \times [z_0 + 2\gamma, 1 - \gamma].$$

Clearly, the probability $\mathbb{P}[\Omega_{a,T}^M]$ does not depend on a nor on λ and π and is positive. We then define $q_T > 0$ by

$$\mathbb{P}[\Omega_{a,T}^M] = 2q_T. \quad (\star)$$

We then use basic considerations on i.i.d. Poisson processes with rate 1 (we write \mathbb{P}_M for the conditional probability w.r.t. π_M) to show that point 2. occurs with high probability.

- For $k = 1, 2$, we have $T_k^\lambda \geq \mathbf{a}_\lambda(z_0 + 2\gamma)$ and

$$\begin{aligned} \mathbb{P}_M \left[\forall i \in \llbracket X_k^\lambda - \mathbf{m}_\lambda^\gamma, X_k^\lambda + \mathbf{m}_\lambda^\gamma \rrbracket, N_{T_k^\lambda}^S(i) > 0 \right] &\geq (1 - \lambda^{z_0+2\gamma})^{2\mathbf{m}_\lambda^\gamma+1} \\ &\simeq \exp(-\lambda^{z_0+2\gamma} \frac{\gamma \lambda^{-\gamma-(1-\gamma)z_0}}{\mathbf{a}_\lambda}) = \exp(-\gamma \frac{\lambda^{\gamma(z_0+1)}}{\mathbf{a}_\lambda}) \end{aligned}$$

which tends to 1 when $\lambda \rightarrow 0$.

- For $k = 1, 2$, we have

$$\mathbb{P}_M \left[\exists i_2^k \in \llbracket X_k^\lambda + \mathbf{m}_\lambda^\gamma + 1, X_k^\lambda + \mathbf{m}_\lambda - 1 \rrbracket, N_{\mathbf{a}_\lambda(1-\gamma)}^S(i_2^k) = 0 \right] = 1 - (1 - \lambda^{1-\gamma})^{\mathbf{m}_\lambda - \mathbf{m}_\lambda^\gamma - 1}$$

which tends to 1 when $\lambda \rightarrow 0$, because $\mathbf{m}_\lambda^\gamma \ll \mathbf{m}_\lambda$ and $\lambda^{1-\gamma} \ll \mathbf{m}_\lambda$ (similar computation holds for i_1^k).

Finally, since π_M is independent of the processes family $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, Lemma 4.2 directly imply that, for all $k = 1, 2$, $\mathbb{P}_M \left[\Omega_{\lambda, \pi}^{P, T}(X_k, T_k) \right]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

All this, together with (\star) , implies that $\mathbb{P} \left[\Omega_{a, T}^{\lambda, \pi} \right] \geq q_T > 0$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

In the end, for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$, the event $\Omega_{a, T}^{\lambda, \pi}$ depend only on the Poisson processes $N_t^S(i)$, $N_t^M(i)$ and $N_t^P(i)$ for $t \in [0, \mathbf{a}_\lambda(T+2)]$ and $i \in J_a^\lambda$. This suffices to conclude the proof in the case $z_0 \in [0, 1)$.

Case 2: $z_0 = 1$. Fix some $\alpha > 0$ small enough, say $\alpha = 0.001$. Recall that

$$\kappa_{\lambda, \pi}^{1-\alpha} = \frac{1}{\lambda^{1-\alpha} \mathbf{a}_\lambda \pi} + \varepsilon_\lambda$$

and assume that $\kappa_{\lambda, \pi}^{1-\alpha} < \alpha$. We first define the event $\tilde{\Omega}_{a, T}^{\lambda, \pi}$ on which points 1 and 2 below are satisfied:

1. The family of Poisson processes $(N_t^M(i))_{t \in [0, \mathbf{a}_\lambda T], i \in J_a^\lambda}$ has exactly 4 marks in J_a^λ , and we call them $(X_k^\lambda, T_k^\lambda)_{k=1, \dots, 4}$, in such a way the match $(X_1^\lambda, T_1^\lambda)$ (resp. $(X_2^\lambda, T_2^\lambda)$) belongs to

$$\begin{aligned} & \llbracket \lfloor (a + \alpha) \mathbf{n}_\lambda \rfloor, \lfloor (a + \frac{1}{4} - \alpha) \mathbf{n}_\lambda \rfloor \rrbracket \times [\mathbf{a}_\lambda(\frac{3}{4} + \alpha), \mathbf{a}_\lambda(1 - 2\alpha)] \\ & \text{(resp. } \llbracket \lfloor (a + \frac{3}{4} + \alpha) \mathbf{n}_\lambda \rfloor, \lfloor (a + 1 - \alpha) \mathbf{n}_\lambda \rfloor \rrbracket \times [\mathbf{a}_\lambda(\frac{3}{4} + \alpha), \mathbf{a}_\lambda(1 - 2\alpha)]), \end{aligned}$$

and the match $(X_3^\lambda, T_3^\lambda)$ (resp. $(X_4^\lambda, T_4^\lambda)$) belongs to

$$\begin{aligned} & \llbracket \lfloor (a + \frac{1}{4} + \alpha) \mathbf{n}_\lambda \rfloor, \lfloor (a + \frac{1}{2} - \alpha) \mathbf{n}_\lambda \rfloor \rrbracket \times [\mathbf{a}_\lambda(1 + \alpha), \mathbf{a}_\lambda(\frac{5}{4} - 2\alpha)] \\ & \text{(resp. } \llbracket \lfloor (a + \frac{1}{2} + \alpha) \mathbf{n}_\lambda \rfloor, \lfloor (a + \frac{3}{4} - \alpha) \mathbf{n}_\lambda \rfloor \rrbracket \times [\mathbf{a}_\lambda(1 + \alpha), \mathbf{a}_\lambda(\frac{5}{4} - 2\alpha)]). \end{aligned}$$

2. The family of Poisson processes $(N_t^S(i))_{t \geq 0, i \in J_a^\lambda}$ satisfies,

- (a) for $k = 1, 2$, $\forall i \in \llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket$, $N_{T_k^\lambda}^S(i) > 0$;
- (b) for $k = 1, 2$, there are $i_1^k \in \llbracket X_k^\lambda - \lfloor \lambda^{-(1-\alpha)} \rfloor - 1, X_k^\lambda \rrbracket$ and $i_2^k \in \llbracket X_k^\lambda, X_k^\lambda + \lfloor \lambda^{-(1-\alpha)} \rfloor + 1 \rrbracket$ such that $N_{T_k^\lambda + \mathbf{a}_\lambda \kappa_{\lambda, \pi}^{1-\alpha}}^S(i_j^k) = 0$.
- (c) for $k = 1, 2$, there exists $i_3^k \in \llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket$ such that $N_{3\mathbf{a}_\lambda/2}^S(i_3^k) - N_{T_k^\lambda}^S(i_3^k) = 0$;
- (d) $\forall i \in \llbracket \lfloor a \mathbf{n}_\lambda \rfloor, \lfloor (a + 1) \mathbf{n}_\lambda \rfloor \rrbracket$, $N_{\mathbf{a}_\lambda(1+\alpha)}^S(i) > 0$.

We now introduce the event on which all these four fires propagate on the good speed

$$\Omega_{a, T}^P(\lambda, \pi) = \Omega_{\lambda, \pi}^{P, 1-\alpha}(\frac{X_1^\lambda}{\mathbf{n}_\lambda}, \frac{T_1^\lambda}{\mathbf{a}_\lambda}) \cap \Omega_{\lambda, \pi}^{P, 1-\alpha}(\frac{X_2^\lambda}{\mathbf{n}_\lambda}, \frac{T_2^\lambda}{\mathbf{a}_\lambda}) \cap \Omega_{\lambda, \pi}^{P, T, \alpha}(\frac{X_3^\lambda}{\mathbf{n}_\lambda}, \frac{T_3^\lambda}{\mathbf{a}_\lambda}) \cap \Omega_{\lambda, \pi}^{P, T, \alpha}(\frac{X_4^\lambda}{\mathbf{n}_\lambda}, \frac{T_4^\lambda}{\mathbf{a}_\lambda}),$$

recall Definition 4.7.

We finally set

$$\Omega_{a, T}^{\lambda, \pi} = \tilde{\Omega}_{a, T}^{\lambda, \pi} \cap \Omega_{a, T}^P(\lambda, \pi).$$

We deduce that $\Omega_{a,T}^{\lambda,\pi}$ satisfies (i) as above: the match falling in X_k^λ , for $k = 1, 2$, destroys at least the zone $\llbracket X_k^\lambda - \lfloor \lambda^{-3/4} \rfloor, X_k^\lambda + \lfloor \lambda^{-3/4} \rfloor \rrbracket$ (thanks to 2-(a)) but does not affect the zone outside $\llbracket X_k^\lambda - \lfloor \lambda^{-(1-\alpha)} \rfloor, X_k^\lambda + \lfloor \lambda^{-(1-\alpha)} \rfloor \rrbracket$ (thanks to 2-(b) and recall **Micro**($\infty, 1$) in Subsection 4.4). Hence, for $k = 1, 2$, i_3^k remains vacant from $T_k^\lambda + \mathbf{a}_\lambda \kappa_{\lambda,\pi}^{1-\alpha}$ until $3\mathbf{a}_\lambda/2$. Thus, i_3^1 and i_3^2 protect the zone $\llbracket \lfloor (a + \frac{1}{4} - \alpha)\mathbf{n}_\lambda \rfloor, \lfloor (a + \frac{3}{4} - \alpha)\mathbf{n}_\lambda \rfloor \rrbracket$, which is completely filled at time $\mathbf{a}_\lambda(1 + \alpha)$, thanks to 2-(d). As previously, and since fires have only a local effect (recall that $\mathbf{m}_\lambda^\alpha = \lfloor \alpha \mathbf{n}_\lambda \rfloor$), the right front of the fire 3 and the left front of the fire 4 burn until $\mathbf{a}_\lambda T$.

We then can set, for all $t \in [0, \mathbf{a}_\lambda T]$

$$\iota_t^+ = \begin{cases} i_1^1 & \text{if } 0 \leq t < T_1^\lambda + \mathbf{a}_\lambda \kappa_{\mathbf{a}_\lambda, \pi}^{1-\alpha}, \\ i_3^1 & \text{if } T_1^\lambda + \mathbf{a}_\lambda \kappa_{\mathbf{a}_\lambda, \pi}^{1-\alpha} \leq t < T_3^\lambda, \\ X_3^\lambda + i_{t-T_3^\lambda}^{3,+} & \text{if } T_3^\lambda \leq t \leq \mathbf{a}_\lambda T, \end{cases}$$

and

$$\iota_t^- = \begin{cases} i_2^2 & \text{if } 0 \leq t < T_2^\lambda + \mathbf{a}_\lambda \kappa_{\mathbf{a}_\lambda, \pi}^{1-\alpha}, \\ i_3^2 & \text{if } T_2^\lambda + \mathbf{a}_\lambda \kappa_{\mathbf{a}_\lambda, \pi}^{1-\alpha} \leq t < T_4^\lambda, \\ X_4^\lambda + i_{t-T_4^\lambda}^{4,-} & \text{if } T_4^\lambda \leq t \leq \mathbf{a}_\lambda T. \end{cases}$$

We can check, as usual, that $\mathbb{P}[\Omega_{a,T}^{\lambda,\pi}] \geq q_T$, for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, 1)$, where $2q_T$ is the probability that a Poisson measure π_M has exactly 4 marks $(X_k, T_k)_{k=1,\dots,4}$ in $[a, a+1] \times [0, T]$ in such a way that

$$\begin{aligned} (X_1, T_1) &\in [a + \alpha, a + \frac{1}{4} - \alpha] \times [\frac{3}{4} + \alpha, 1 - 2\alpha], \\ (X_2, T_2) &\in [a + \frac{3}{4} + \alpha, a + 1 - \alpha] \times [\frac{3}{4} + \alpha, 1 - 2\alpha], \\ (X_3, T_3) &\in [a + \frac{1}{4} + \alpha, a + \frac{1}{2} - \alpha] \times [1 + \alpha, \frac{5}{4} - 2\alpha], \\ (X_4, T_4) &\in [a + \frac{1}{2} + \alpha, a + \frac{3}{4} - \alpha] \times [1 + \alpha, \frac{5}{4} - 2\alpha]. \end{aligned} \quad \square$$

Proof in the regime $\mathcal{R}(0)$. We fix $T > 0$. It of course suffices to prove the result for A large enough. We consider the true (λ, π) -FFP $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and set $K = \lfloor 4T \rfloor$. For $a \in \mathbb{R}$, we recall that

$$\varkappa_{\lambda,\pi} = \varkappa_{\lambda,\pi}^{2K} = \frac{2K\mathbf{n}_\lambda}{\mathbf{a}_\lambda\pi} + \varepsilon_\lambda$$

and

$$J_a^\lambda := \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor (a+1)\mathbf{n}_\lambda \rfloor - 1 \rrbracket$$

and introduce

$$J_{a,K}^\lambda := \llbracket \lfloor (a-3K)\mathbf{n}_\lambda \rfloor, \lfloor (a+3K+1)\mathbf{n}_\lambda \rfloor - 1 \rrbracket.$$

As usual, for $a \in \mathbb{R}$, we are going to build an event $\Omega_{a,T}^{\lambda,\pi}$ depending only on the Poisson processes $N_t^S(i)$, $N_t^M(i)$ and $N_t^P(i)$ for $t \in [0, \mathbf{a}_\lambda T]$ and $i \in J_{a,K}^\lambda$ such that

- (i) on $\Omega_{a,T}^{\lambda,\pi}$, there exists $\iota^+ : [0, \mathbf{a}_\lambda T] \mapsto J_{a,K}^\lambda$ (resp. $\iota^- : [0, \mathbf{a}_\lambda T] \mapsto J_{a,K}^\lambda$), non decreasing (resp. non increasing), such that $\eta_t^{\lambda,\pi}(\iota_t^+) = 0$ (resp. $\eta_t^{\lambda,\pi}(\iota_t^-) = 0$) for all $t \in [0, \mathbf{a}_\lambda T]$,
- (ii) there exists $q_T > 0$ such that for all $a \in \mathbb{R}$, we have $\mathbb{P}[\Omega_{a,T}^{\lambda,\pi}] \geq q_T$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

It is then routine to conclude the proof. We fix $\alpha = 0.001$ and assume that (λ, π) is sufficiently close to the regime $\mathcal{R}(0)$ in such a way that $\varkappa_{\lambda,\pi} \leq \alpha$.

Step 1. Here we show that for all $b \in \mathbb{R}$, there exists an event $\Omega_{b,0}^{\lambda,\pi}$, depending only on $(N_s^S(i), N_s^M(i), N_s^P(i))_{s \in [0, 3\mathbf{a}_\lambda/4], i \in J_b^\lambda}$ such that

(i) on $\Omega_{b,0}^{\lambda,\pi}$, a.s., there is $i \in J_b^\lambda$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(i) = 0$ for all $s \in [0, 3/4]$;

(ii) $\lim_{\lambda \rightarrow 0} \mathbb{P} \left[\Omega_{b,0}^{\lambda,\pi} \right] = 1$.

Simply consider the event $\Omega_{b,0}^{\lambda,\pi} = \{\exists i \in J_b^\lambda, N_{3\mathbf{a}_\lambda/4}^S(i) = 0\}$. Clearly, point (i) is satisfied, since there is a site in J_b^λ on which no seed falls during $[0, 3\mathbf{a}_\lambda/4]$. Since $|J_b^\lambda| \simeq \mathbf{n}_\lambda \simeq 1/(\lambda \log(1/\lambda))$, we deduce that

$$\mathbb{P} \left[\Omega_{b,0}^{\lambda,\pi} \right] = 1 - (1 - e^{-3\mathbf{a}_\lambda/4})^{\mathbf{n}_\lambda} \simeq 1 - e^{-1/(\lambda^{1/4}\mathbf{a}_\lambda)} \xrightarrow{\lambda \rightarrow 0} 1,$$

whence (ii).

Step 2. For $\lambda > 0$ and $\pi \geq 1$, we put $\mathbf{k}_\lambda := \lfloor \lambda^{-3/8} \rfloor$ and observe that $\mathbf{k}_\lambda \ll \mathbf{n}_\lambda$. For $k \in \{1, \dots, K-1\}$, we set

$$\tau_k = \frac{k+1}{4} \text{ and } \tilde{\tau}_k = \frac{k+1}{4} + \frac{1}{8}.$$

Consider the event $\tilde{\Omega}_{a,T}^{\lambda,\pi}$ on which points 1, 2 and 3 below are satisfied.

1. The family of Poisson processes $(N_t^M(i))_{t \in [0, \mathbf{a}_\lambda T], i \in J_{a,K}^\lambda}$ has exactly $2(K-1)$ marks in $J_{a,K}^\lambda$, and we call them

$$\{(X_1^\lambda, T_1^\lambda), \dots, (X_{K-1}^\lambda, T_{K-1}^\lambda)\} \text{ and } \{(\tilde{X}_1^\lambda, \tilde{T}_1^\lambda), \dots, (\tilde{X}_{K-1}^\lambda, \tilde{T}_{K-1}^\lambda)\},$$

in such a way that, for all $k \in \{1, \dots, K-1\}$,

$$(X_k^\lambda, T_k^\lambda) \in \llbracket \lfloor (a - K + k + \frac{1}{3})\mathbf{n}_\lambda \rfloor, \lfloor (a - K + k + \frac{2}{3})\mathbf{n}_\lambda \rfloor \rrbracket \times [(\tau_k - 1/12)\mathbf{a}_\lambda, (\tau_k - \varkappa_{\lambda,\pi})\mathbf{a}_\lambda]$$

and

$$\begin{aligned} (\tilde{X}_k^\lambda, \tilde{T}_k^\lambda) \in \llbracket \lfloor (a + K - (k+1) + \frac{1}{3})\mathbf{n}_\lambda \rfloor, \lfloor (a + K - (k+1) + \frac{2}{3})\mathbf{n}_\lambda \rfloor \rrbracket \\ \times [(\tilde{\tau}_k - 1/12)\mathbf{a}_\lambda, (\tilde{\tau}_k - \varkappa_{\lambda,\pi})\mathbf{a}_\lambda]. \end{aligned}$$

(See Figure 4 for a graphical example.)

2. The family of Poisson processes $(N_t^S(i))_{t \geq 0, i \in J_{a,K}^\lambda}$ satisfies, for all $k \in \{1, \dots, K-1\}$,
 - (a) there are $j_g \in \llbracket \lfloor (a - K + k)\mathbf{n}_\lambda \rfloor, \lfloor (a - K + k + 1/4)\mathbf{n}_\lambda \rfloor \rrbracket$ and $j_d \in \llbracket \lfloor (a - K + k + 3/4)\mathbf{n}_\lambda \rfloor, \lfloor (a - K + k + 1)\mathbf{n}_\lambda - 1 \rfloor \rrbracket$ such that

$$N_{\mathbf{a}_\lambda(\tau_k+1/4)}^S(j_g) - N_{\mathbf{a}_\lambda(\tau_k-1/2)}^S(j_g) = N_{\mathbf{a}_\lambda(\tau_k+1/4)}^S(j_d) - N_{\mathbf{a}_\lambda(\tau_k-1/2)}^S(j_d) = 0;$$

- (b) for all $i \in \llbracket X_k^\lambda - \mathbf{k}_\lambda, X_k^\lambda + \mathbf{k}_\lambda \rrbracket$,

$$N_{\mathbf{a}_\lambda(\tau_k-1/12)}^S(i) - N_{\mathbf{a}_\lambda(\tau_k-1/2)}^S(i) > 0;$$

- (c) there is $j_0 \in \llbracket X_k^\lambda - \mathbf{k}_\lambda, X_k^\lambda + \mathbf{k}_\lambda \rrbracket$ such that

$$N_{\mathbf{a}_\lambda(\tau_k+1/4)}^S(j_0) - N_{\mathbf{a}_\lambda(\tau_k-1/12)}^S(j_0) = 0.$$

3. The family of Poisson processes $(N_t^S(i))_{t \geq 0, i \in J_{a,K}^\lambda}$ satisfies, for all $k \in \{1, \dots, K-1\}$,

- (a) there are $j_g \in \llbracket \lfloor (a + K - (k+1))\mathbf{n}_\lambda \rfloor, \lfloor (a + K - (k+1) + 1/4)\mathbf{n}_\lambda \rfloor \rrbracket$ and $j_d \in \llbracket \lfloor (a + K - (k+1) + 3/4)\mathbf{n}_\lambda \rfloor, \lfloor (a + K - (k+1) + 1)\mathbf{n}_\lambda - 1 \rfloor \rrbracket$ such that

$$N_{\mathbf{a}_\lambda(\tilde{\tau}_k+1/4)}^S(j_g) - N_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/2)}^S(j_g) = N_{\mathbf{a}_\lambda(\tilde{\tau}_k+1/4)}^S(j_d) - N_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/2)}^S(j_d) = 0;$$

- (b) for all $i \in \llbracket \tilde{X}_k^\lambda - \mathbf{k}_\lambda, \tilde{X}_k^\lambda + \mathbf{k}_\lambda \rrbracket$,

$$N_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/12)}^S(i) - N_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/2)}^S(i) > 0;$$

(c) there is $j_0 \in \llbracket \tilde{X}_k^\lambda - \mathbf{k}_\lambda, \tilde{X}_k^\lambda + \mathbf{k}_\lambda \rrbracket$ such that

$$N_{\mathbf{a}_\lambda(\tilde{\tau}_k+1/4)}^S(j_0) - N_{\mathbf{a}_\lambda(\tilde{\tau}_k-1/12)}^S(j_0) = 0.$$

We also introduce the event

$$\Omega_{\lambda,\pi}^{P,K} = \left(\bigcap_{k=1}^{K-1} \Omega_{\lambda,\pi}^{P,2K,2K} \left(\frac{X_k^\lambda}{\mathbf{n}_\lambda}, \frac{T_k^\lambda}{\mathbf{a}_\lambda} \right) \right) \cap \left(\bigcap_{k=1}^{K-1} \Omega_{\lambda,\pi}^{P,2K,2K} \left(\frac{\tilde{X}_k^\lambda}{\mathbf{n}_\lambda}, \frac{\tilde{T}_k^\lambda}{\mathbf{a}_\lambda} \right) \right),$$

recall Definition 4.7.

Finally, we set

$$\Omega_{a,T}^{\lambda,\pi} = \tilde{\Omega}_{a,T}^{\lambda,\pi} \cap \Omega_{\lambda,\pi}^{P,K} \cap \Omega_{a-K,0}^{\lambda,\pi} \cap \Omega_{a+K-1,0}^{\lambda,\pi}.$$

Step 3. Here we prove (ii).

The probability of the event on which N^M satisfies 1. does not depend on $a \in \mathbb{R}$ by invariance by spatial translation. We also can construct N^M using a Poisson measure π_M on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$, independent of N^S and N^P , by setting, for all $i \in \mathbb{Z}$

$$N_t^M(i) = \pi_M(i_\lambda \times [0, t/\mathbf{a}_\lambda]).$$

As usual, for all $\lambda > 0$ small enough, the probability of the event on which N^M satisfies 1 is then bounded from below by some constant $2q_T > 0$, which does not depend on $a \in \mathbb{R}$ nor on $\lambda > 0$ and $\pi \geq 1$. We write \mathbb{P}_M for the conditional probability w.r.t. π_M .

Let now $k \in \{1, \dots, K-1\}$. The probability of 2-(a) tends to 1. Indeed, treating e.g. the case of j_g , there holds, recalling $\mathbf{n}_\lambda \simeq 1/(\lambda \mathbf{a}_\lambda)$ and $\mathbf{a}_\lambda = \log(1/\lambda)$,

$$\begin{aligned} \mathbb{P} \left[\exists j \in \llbracket (a-K+k)\mathbf{n}_\lambda, (a-K+k+1/4)\mathbf{n}_\lambda \rrbracket, N_{\mathbf{a}_\lambda(\tau_k+1/4)}^S(j) - N_{\mathbf{a}_\lambda(\tau_k-1/2)}^S(j) = 0 \right] \\ = 1 - (1 - e^{-(3/4)\mathbf{a}_\lambda})^{\mathbf{n}_\lambda/4} \simeq 1 - e^{-\mathbf{n}_\lambda \lambda^{3/4}/4} \xrightarrow{\lambda \rightarrow 0} 1. \end{aligned}$$

The probability of 2-(b) (conditionally on π_M) also tends to 1. Indeed, it equals

$$(1 - e^{-5\mathbf{a}_\lambda/12})^{2\mathbf{k}_\lambda+1} \simeq e^{-2\mathbf{k}_\lambda \lambda^{5/12}} \xrightarrow{\lambda \rightarrow 0} 1$$

since $\mathbf{k}_\lambda = \lfloor \lambda^{-3/8} \rfloor$ and since $3/8 < 5/12$. Finally, the probability of 2-(c) (conditionally on π_M) also tends to 1, since it equals

$$1 - (1 - e^{-\mathbf{a}_\lambda/3})^{2\mathbf{k}_\lambda+1} \simeq 1 - e^{-2\mathbf{k}_\lambda \lambda^{1/3}}$$

which tends to 1 when $\lambda \rightarrow 0$, since $1/3 < 3/8$.

Similar considerations hold for Point 3.

Finally, since π_M is independent of the processes family $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, Lemma 4.3 directly implies that, using space/time stationarity, for all $k \in \{1, \dots, K-1\}$,

$$\mathbb{P}_M \left[\Omega_{\lambda,\pi}^{P,2K,2K} (X_k^\lambda/\mathbf{n}_\lambda, T_k^\lambda/\mathbf{a}_\lambda) \right] = \mathbb{P}_M \left[\Omega_{\lambda,\pi}^{P,2K,2K} (\tilde{X}_k^\lambda/\mathbf{n}_\lambda, \tilde{T}_k^\lambda/\mathbf{a}_\lambda) \right]$$

tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

All this implies that there exists $q_T > 0$ such that $\mathbb{P} \left[\Omega_{a,T}^{\lambda,\pi} \right] > q_T$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

Step 4. Here we work on $\Omega_{a,T}^{\lambda,\pi}$ and we prove that, for all $k \in \{1, \dots, K-1\}$, if there is no burning tree in J_{a-K+k}^λ at time $(\tau_k - 1/2)\mathbf{a}_\lambda$, then there is $i \in J_{a-K+k}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i) = 0$ for all $t \in [\tau_k, \tau_k + 1/4]$. We distinguish two cases.

- If the zone $\llbracket X_k^\lambda - \mathbf{k}_\lambda, X_k^\lambda + \mathbf{k}_\lambda \rrbracket$ is completely occupied at time $T_k^\lambda -$, then each site burns at least one time (i.e. each site in this zone is ignited and then extinguished) during $[T_k^\lambda, T_k^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}]$, thanks to $\Omega_{\lambda, \pi}^{P, 2K, 2K}(X_k^\lambda / \mathbf{n}_\lambda, T_k^\lambda / \mathbf{a}_\lambda)$, recall **Macro**(0) in Subsection 4.4. Since no seed falls on j_0 , which belongs to this zone, during

$$[\mathbf{a}_\lambda(\tau_k - 1/12), \mathbf{a}_\lambda(\tau_k + 1/4)] \supset [T_k^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}, \mathbf{a}_\lambda(\tau_k + 1/4)] \supset [\mathbf{a}_\lambda \tau_k, \mathbf{a}_\lambda(\tau_k + 1/4)],$$

we deduce that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(j_0) = 0$ for all $s \in [\tau_k, \tau_k + 1/4]$.

- Assume now that there exists $i_0 \in \llbracket X_k^\lambda - \mathbf{k}_\lambda, X_k^\lambda + \mathbf{k}_\lambda \rrbracket$ that is vacant at time $T_k^\lambda -$. Recall that there is no match falling in J_a^λ during $[\mathbf{a}_\lambda(\tau_k - 1/2), T_k^\lambda]$, that on each site of $\llbracket X_k^\lambda - \mathbf{k}_\lambda, X_k^\lambda + \mathbf{k}_\lambda \rrbracket$, at least one seed falls during $[\mathbf{a}_\lambda(\tau_k - 1/2), \mathbf{a}_\lambda(\tau_k - 1/12)] \subset [\mathbf{a}_\lambda(\tau_k - 1/2), T_k^\lambda]$ and that there is no burning tree in J_{a-K+k}^λ at time $\mathbf{a}_\lambda(\tau_k - 1/2)$. Then necessarily, a fire starting at some $i'_M \notin J_{a-K+k}^\lambda$ at some time $t'_M < T_k^\lambda$, has made vacant i_0 . Assume e.g. that $i'_M < \lfloor (a - K + k)\mathbf{n}_\lambda \rfloor$ and observe that $i'_M < j_g < i_0$. The fire (i'_M, t'_M) has then also necessarily made vacant j_g during $(\mathbf{a}_\lambda(\tau_k - 1/2), T_k^\lambda)$. Since no seed falls on j_g during $[\mathbf{a}_\lambda(\tau_k - 1/2), \mathbf{a}_\lambda(\tau_k + 1/4)]$, we deduce that j_g remains vacant during $[\mathbf{a}_\lambda \tau_k, \mathbf{a}_\lambda(\tau_k + 1/4)]$.

Step 5. We can show, exactly as above, that, on $\Omega_{a, T}^{\lambda, \pi}$, if there is no burning tree in $J_{a+K-(k+1)}^\lambda$ at time $(\tilde{\tau}_k - 1/2)\mathbf{a}_\lambda$, for some $k \in \{1, \dots, K-1\}$, then there is $i \in J_{a+K-(k+1)}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 0$ for all $t \in [\tilde{\tau}_k, \tilde{\tau}_k + 1/4]$.

Step 6. To conclude the proof, we now prove by induction (see Figure 4) that for all $k \in \{1, \dots, K-1\}$

- there exists $i_k \in J_{a-K+k}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i_k) = 0$ for all $t \in [\tau_k, \tau_k + 1/4]$;
- there exists $j_k \in J_{a+K-(k+1)}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(j_k) = 0$ for all $t \in [\tilde{\tau}_k, \tilde{\tau}_k + 1/4]$;
- there is no burning tree in $\llbracket i_k, j_k \rrbracket$ at time $\mathbf{a}_\lambda \tau_k$ nor at time $\mathbf{a}_\lambda \tilde{\tau}_k$.

▷ At time 0, all sites are vacant. Thus, there are $i_0 \in J_{a-K}^\lambda$ and $j_0 \in J_{a+K-1}^\lambda$ which remain vacant until time $3\mathbf{a}_\lambda/4$ (thanks to $\Omega_{a-K, 0}^{\lambda, \pi} \cap \Omega_{a+K-1, 0}^{\lambda, \pi}$). Since no match falls in $\llbracket i_0, j_0 \rrbracket$ until time $T_1^\lambda \geq \mathbf{a}_\lambda(1/2 - 1/12) = 5\mathbf{a}_\lambda/12$, there is no burning tree in $[0, 5\mathbf{a}_\lambda/12]$ (no match falling outside $\llbracket i_0, j_0 \rrbracket$ during $[0, 5\mathbf{a}_\lambda/12]$ can affect this zone).

Thus, Step 4 shows that there are $i_1 \in J_{a-K+1}^\lambda$ which is vacant during $[\mathbf{a}_\lambda/2, 3\mathbf{a}_\lambda/4]$ (because $\tau_1 - 1/2 = 0$) and $i_2 \in J_{a-K+2}^\lambda$ which is vacant during $[3\mathbf{a}_\lambda/4, \mathbf{a}_\lambda]$ (because $\tau_2 - 1/2 = 1/4 < 5/12$). Similarly, Step 5 above shows that there are $j_1 \in J_{a+K-2}^\lambda$ which is vacant during $[5\mathbf{a}_\lambda/8, 7\mathbf{a}_\lambda/8]$ (because $\tilde{\tau}_1 - 1/2 = 1/8 < 5/12$) and $j_2 \in J_{a+K-3}^\lambda$ which is vacant during $[7\mathbf{a}_\lambda/8, 9\mathbf{a}_\lambda/8]$ (because $\tilde{\tau}_2 - 1/2 = 3/8 < 5/12$).

Since $T_1^\lambda \leq (1/2 - \varkappa_{\lambda, \pi})\mathbf{a}_\lambda$ and $|X_1^\lambda - i_0| \leq |X_1^\lambda - j_0| \leq 2K\mathbf{n}_\lambda$, as seen in **Macro**(0) in Subsection 4.4 (recall that we work on $\Omega_{\lambda, \pi}^{P, 2K, 2K}(X_1^\lambda / \mathbf{n}_\lambda, T_1^\lambda / \mathbf{a}_\lambda)$), there is no more burning tree in $\llbracket i_0, j_0 \rrbracket$ at time $T_1^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi} \leq \mathbf{a}_\lambda/2 = \mathbf{a}_\lambda \tau_1$. Since no match falls in $\llbracket i_0, j_0 \rrbracket$ during $[T_1^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}, \mathbf{a}_\lambda/2]$, we deduce that there is also no burning tree in $\llbracket i_0, j_0 \rrbracket \supset \llbracket i_1, j_1 \rrbracket$ at time $\mathbf{a}_\lambda \tau_1$ (because i_0 and j_0 remain vacant until $\mathbf{a}_\lambda/2$).

Since no match falls in $\llbracket i_1, j_0 \rrbracket$ during $[\mathbf{a}_\lambda \tau_1, \tilde{T}_1^\lambda]$, we deduce that there is no burning tree in $\llbracket i_1, j_0 \rrbracket$ at time $\tilde{T}_1^\lambda -$. Since $\eta_{\tilde{T}_1^\lambda}^{\lambda, \pi}(i_1) = \eta_{\tilde{T}_1^\lambda}^{\lambda, \pi}(j_0) = 0$ for all $t \in [\tilde{T}_1^\lambda, \tilde{T}_1^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}]$ and only one match falls in $\llbracket i_1, j_0 \rrbracket$ during $[\tilde{T}_1^\lambda, \tilde{T}_1^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}]$, we deduce, recall **Macro**(0) in Subsection 4.4, that there is no more burning tree in $\llbracket i_1, j_0 \rrbracket$ at time $\tilde{T}_1^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}$. We easily deduce that there is also no burning tree in $\llbracket i_1, j_1 \rrbracket \subset \llbracket i_1, j_0 \rrbracket$ at time $\mathbf{a}_\lambda \tilde{\tau}_1$.

Similarly, since $i_0 < i_1 < i_2 < j_2 < j_1 < j_0$ and thanks to $\Omega_{\lambda, \pi}^{P, K}$, there is no more burning tree in $\llbracket i_1, j_1 \rrbracket \supset \llbracket i_2, j_2 \rrbracket$ at time τ_2 nor in $\llbracket i_2, j_1 \rrbracket \supset \llbracket i_2, j_2 \rrbracket$ at time $\tilde{\tau}_2$.

▷ Assume now that there is $k \in \{2, \dots, K-2\}$ such that, for all $l \leq k$,

- there exists $i_l \in J_{a-K+l}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i_l) = 0$ for all $t \in [\mathbf{a}_\lambda \tau_l, \mathbf{a}_\lambda(\tau_l + 1/4)]$;

- there exists $j_l \in J_{a+K-(l+1)}^\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(j_l) = 0$ for all $t \in [\mathbf{a}_\lambda \tilde{\tau}_l, \mathbf{a}_\lambda(\tilde{\tau}_l + 1/4)]$;
- there is no burning tree in $\llbracket i_l, j_l \rrbracket$ at time $\mathbf{a}_\lambda \tilde{\tau}_l$ nor at time $\mathbf{a}_\lambda \tilde{\tau}_l$.

Since there is no burning tree in $J_{a-K+k+1}^\lambda \subset \llbracket i_{k-1}, j_{k-1} \rrbracket$ at time $\mathbf{a}_\lambda \tau_{k-1} = \mathbf{a}_\lambda(\tau_{k+1} - 1/2)$, see Step 4, there is $i_{k+1} \in J_{a-K+k+1}^\lambda$ which is vacant during $[\mathbf{a}_\lambda \tau_{k+1}, \mathbf{a}_\lambda(\tau_{k+1} + 1/4)]$. Furthermore, no match falls in $\llbracket i_k, j_k \rrbracket$ during $[\mathbf{a}_\lambda \tilde{\tau}_k, T_{k+1}^\lambda] \subset [\mathbf{a}_\lambda \tilde{\tau}_k, \mathbf{a}_\lambda(\tau_{k+1} - \varkappa_{\lambda, \pi})]$ and there is no burning tree in $\llbracket i_k, j_k \rrbracket$ at time $\mathbf{a}_\lambda \tilde{\tau}_k$, thus, as seen in **Macro**(0) in Subsection 4.4 and thanks to $\Omega_{\lambda, \pi}^{P, 2K, 2K} \left(\frac{X_{k+1}^\lambda}{\mathbf{n}_\lambda}, \frac{T_{k+1}^\lambda}{\mathbf{a}_\lambda} \right)$, there is no more burning tree in $\llbracket i_k, j_k \rrbracket$ at time $T_{k+1}^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}$ nor at time $\mathbf{a}_\lambda \tau_{k+1}$ (because i_k and j_k remain vacant until $\mathbf{a}_\lambda \tau_{k+1}$ and no match falls in $\llbracket i_k, j_k \rrbracket$ during $(T_{k+1}^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}, \mathbf{a}_\lambda \tau_{k+1}]$).

Since there is no burning tree in $J_{a+K-(k+2)}^\lambda \subset \llbracket i_{k-1}, j_{k-1} \rrbracket$ at time $\mathbf{a}_\lambda \tilde{\tau}_{k-1} = \mathbf{a}_\lambda(\tilde{\tau}_{k+1} - 1/2)$, we deduce by Step 5 that there is $j_{k+1} \in J_{a+K-(k+2)}^\lambda$ which is vacant during $[\mathbf{a}_\lambda \tilde{\tau}_{k+1}, \mathbf{a}_\lambda(\tilde{\tau}_{k+1} + 1/4)]$. No match falls in $\llbracket i_{k+1}, j_k \rrbracket$ during $[\mathbf{a}_\lambda \tau_{k+1}, \tilde{T}_{k+1}^\lambda] \subset [\mathbf{a}_\lambda \tau_{k+1}, \mathbf{a}_\lambda(\tilde{\tau}_{k+1} - \varkappa_{\lambda, \pi})]$ and there is no burning tree in $\llbracket i_{k+1}, j_k \rrbracket$ at time $\mathbf{a}_\lambda \tau_{k+1}$, thus, as seen in **Macro**(0) in Subsection 4.4 and thanks to $\Omega_{\lambda, \pi}^{P, 2K, 2K} \left(\frac{\tilde{X}_{k+1}^\lambda}{\mathbf{n}_\lambda}, \frac{\tilde{T}_{k+1}^\lambda}{\mathbf{a}_\lambda} \right)$, there is no more burning tree in $\llbracket i_{k+1}, j_k \rrbracket$ at time $\tilde{T}_{k+1}^\lambda + \mathbf{a}_\lambda \varkappa_{\lambda, \pi}$ nor at time $\mathbf{a}_\lambda \tilde{\tau}_{k+1}$, as usual.

By the induction above, we deduce that there are

$$\iota^+ : [0, T] \rightarrow J_{a, K}^\lambda$$

non decreasing, such that for all $t \in [0, T]$, $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\iota_{\mathbf{a}_\lambda t}^+) = 0$ and

$$\iota^- : [0, T] \rightarrow J_{a, K}^\lambda$$

non increasing, such that for all $t \in [0, T]$, $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\iota_{\mathbf{a}_\lambda t}^-) = 0$. This together with Step 3 conclude the proof in the regime $\mathcal{R}(0)$. \square

6 Localization of the result

In this section, we localize Theorems 2.5 and 2.11.

6.1 Localization in the regime $\mathcal{R}(p)$

The following Theorem will be proved in Section 8.

Theorem 6.1. *Let $A > 0$ and $p \geq 0$ be fixed. Consider for each $\lambda \in (0, 1], \pi \geq 1$, the process $(Z_t^{\lambda, \pi, A}(x), D_t^{\lambda, \pi, A}(x))_{t \geq 0, x \in \mathbb{R}}$ associated with the (λ, π, A) -FFP. Consider also the A -LFFP(p) $(Z_t^A(x), H_t^A(x), F_t^A(x))_{t \geq 0, x \in \mathbb{R}}$ and the associated $(D_t^A(x))_{t \geq 0, x \in \mathbb{R}}$. We assume that $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, for some $p \in [0, +\infty)$.*

1. *For any $T > 0$, any finite subset $\{x_1, \dots, x_q\} \subset \mathbb{R}$, $(Z_t^{\lambda, \pi, A}(x_i), D_t^{\lambda, \pi, A}(x_i))_{t \in [0, T], i=1, \dots, q}$ goes in law to $(Z_t^A(x_i), D_t^A(x_i))_{t \in [0, T], i=1, \dots, q}$ in $\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))$. Here $\mathbb{D}([0, T], \mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))$ is endowed with the distance \mathbf{d}_T .*
2. *For any subset $\{(x_1, t_1), \dots, (x_q, t_q)\} \subset \mathbb{R} \times [0, \infty)$, $(Z_{t_i}^{\lambda, \pi, A}(x_i), D_{t_i}^{\lambda, \pi, A}(x_i))_{i=1, \dots, q}$ goes in law to $(Z_{t_i}^A(x_i), D_{t_i}^A(x_i))_{i=1, \dots, q}$ in $(\mathbb{R} \times (\mathcal{I} \cup \{\emptyset\}))^q$. Here $\mathcal{I} \cup \{\emptyset\}$ is endowed with δ .*
3. *For all $t > 0$,*

$$\left(\frac{\log(|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}, 0)| \geq 1\}} \right) \wedge 1$$

goes in law to $Z_t^A(0)$.

Assuming for a moment that this theorem holds true, we conclude the proof of Theorem 2.5.

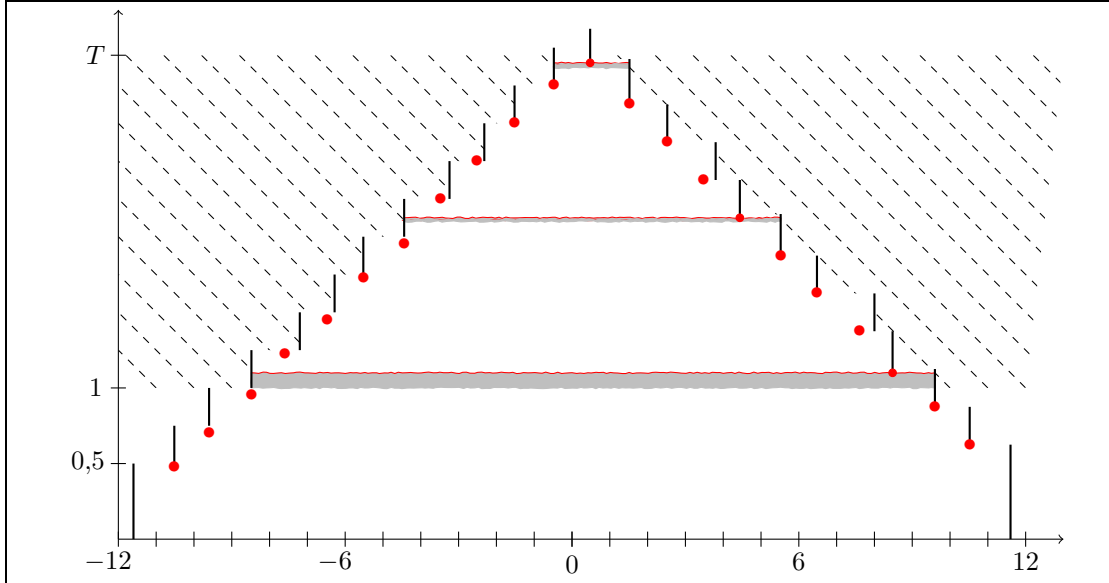


Figure 4: The sweet event

Here $T = 3.2$, $K = 12$ and $a \in [0, 1)$. The marks of π_M (matches) are represented as \bullet 's. The filled zones represent macroscopic zones ($Z_{\mathbf{a}_\lambda t}^{\lambda, \pi}(x) = 1$). In the rest of the space, we always have $Z_{\mathbf{a}_\lambda t}^{\lambda, \pi}(x) < 1$. The plain vertical segments represent vacant sites i.e. sites where no seed falls after being propagated. Remark that sometimes the vacant site is above the match (that is in an interval with length $2\mathbf{k}_\lambda$) and sometimes it is next to the match (that is an i^g or an i^d).

Proof of Theorem 2.5. Let us first prove 1. Consider a continuous bounded function $\Psi : \mathbb{D}([0, T], \mathbb{R} \times \mathcal{I} \cup \{\emptyset\})^q \mapsto \mathbb{R}$. We have to prove that $G_{\lambda, \pi}(\Psi)$ tends to 0 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, where

$$G_{\lambda, \pi}(\Psi) = \mathbb{E} \left[\Psi \left((Z_t^{\lambda, \pi}(x_i), D_t^{\lambda, \pi}(x_i))_{t \in [0, T], i=1, \dots, q} \right) \right] - \mathbb{E} \left[\Psi \left((Z_t(x_i), D_t(x_i))_{t \in [0, T], i=1, \dots, q} \right) \right].$$

Using now Propositions 3.5 and 5.2, we observe that for any $A > 2 \max_{i=1, \dots, q} |x_i|$, there holds that for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$,

$$\begin{aligned} & |G_{\lambda, \pi}(\Psi)| \\ & \leq 2 \|\Psi\|_\infty \mathbb{P} \left[(Z_t^{\lambda, \pi, A}(x), D_t^{\lambda, \pi, A}(x))_{t \in [0, T], x \in [-A/2, A/2]} \neq (Z_t^{\lambda, \pi}(x), D_t^{\lambda, \pi}(x))_{t \in [0, T], x \in [-A/2, A/2]} \right] \\ & \quad + 2 \|\Psi\|_\infty \mathbb{P} \left[(Z_t^A(x), D_t^A(x))_{t \in [0, T], x \in [-A/2, A/2]} \neq (Z_t(x), D_t(x))_{t \in [0, T], x \in [-A/2, A/2]} \right] \\ & \quad + \left| \mathbb{E} \left[\Psi \left((Z_t^{\lambda, \pi, A}(x_i), D_t^{\lambda, \pi, A}(x_i))_{t \in [0, T], i=1, \dots, q} \right) \right] - \mathbb{E} \left[\Psi \left((Z_t^A(x_i), D_t^A(x_i))_{t \in [0, T], i=1, \dots, q} \right) \right] \right| \\ & \leq 4 \|\Psi\|_\infty C_T e^{-\alpha_T A} \\ & \quad + \left| \mathbb{E} \left[\Psi \left((Z_t^{\lambda, \pi, A}(x_i), D_t^{\lambda, \pi, A}(x_i))_{t \in [0, T], i=1, \dots, q} \right) \right] - \mathbb{E} \left[\Psi \left((Z_t^A(x_i), D_t^A(x_i))_{t \in [0, T], i=1, \dots, q} \right) \right] \right|. \end{aligned}$$

Thus Proposition 6.1-(1) implies that $|G_{\lambda, \pi}(\Psi)| \leq 5 \|\Psi\|_\infty C_T e^{-\alpha_T A}$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$. We conclude by making A tend to infinity.

Point (2) is checked similarly. The proof of (3) is also similar, since $D_t^{\lambda, \pi}(0) = D_t^{\lambda, \pi, A}(0)$ implies that $C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0) = C_A(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}, 0)$. \square

6.2 Localization in the regime $\mathcal{R}(\infty, z_0)$

The following Theorem will be proved in the next Section.

Theorem 6.2. Let $z_0 \in [0, 1]$ and $A > 0$. Consider for each $\lambda \in (0, 1]$ and $\pi \geq 1$ the process $(D_t^{\lambda, \pi, A}(x))_{t \geq 0, x \in \mathbb{R}}$ associated with the (λ, π, A) -FFP. Consider also the LFFP (∞, z_0) $(Y_t(x))_{t \geq 0, x \in \mathbb{R}}$ and the associated $(D_t^A(x))_{t \geq 0, x \in \mathbb{R}}$ process. We assume that $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the slow regime $\mathcal{R}(\infty, z_0)$.

1. For any $T > 0$, any finite subset $\{x_1, \dots, x_q\} \subset \mathbb{R}$, $(D_t^{\lambda, \pi, A}(x_i))_{t \in [0, T], i=1, \dots, q}$ goes in law to $(D_t^A(x_i))_{t \in [0, T], i=1, \dots, q}$ in $\mathbb{D}([0, T], \mathcal{I})^q$. Here $\mathbb{D}([0, T], \mathcal{I})^q$ is endowed with δ_T .
2. For any finite subset $\{(x_1, t_1), \dots, (x_q, t_q)\} \subset \mathbb{R} \times [0, \infty)$, $(D_{t_i}^{\lambda, \pi, A}(x_i))_{i=1, \dots, q}$ goes in law to $(D_{t_i}^A(x_i))_{i=1, \dots, q}$ in \mathcal{I}^q , \mathcal{I} being endowed with δ .

Proof of Theorem 2.11. The proof easily follows from Proposition 3.1, Proposition 5.2 and Theorem 6.2, as in the proof above. \square

7 Convergence in the regime $\mathcal{R}(\infty, z_0)$

The aim of this section is to prove Theorem 6.2. We thus fix the parameters $A > 0$ and $T > 0$.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$, $\mathbf{m}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda^2) \rfloor$, $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$ and that

$$A_\lambda = \lfloor A \mathbf{n}_\lambda \rfloor, \\ I_A^\lambda = \llbracket -A_\lambda, A_\lambda \rrbracket.$$

For $x \in \mathbb{R}$, we define

$$(x)_\lambda = \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda \rrbracket.$$

For $\alpha \in (0, 1)$, we also define

$$\mathbf{m}_\lambda^\alpha = \left\lfloor \frac{\alpha}{\lambda^{\alpha+(1-\alpha)z_0} \mathbf{a}_\lambda} \right\rfloor, \\ (x)_\lambda^\alpha = \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda^\alpha, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda^\alpha \rrbracket.$$

Observe that $\mathbf{m}_\lambda^\alpha \leq \lfloor \alpha \mathbf{n}_\lambda \rfloor$ for all $z_0 \in [0, 1]$.

7.1 Occupation of vacant zone

We start with some easy estimates.

Lemma 7.1. Consider a family of i.i.d. Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$. Let $0 < z < 1$, $\alpha \in (0, 1)$ and $a < b$.

1. For $t < z$, $\mathbb{P}[\forall i \in \llbracket \lfloor a\lambda^{-z} \rfloor, \lfloor b\lambda^{-z} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \xrightarrow{\lambda \rightarrow 0} 0$.
2. For $t > z$, $\mathbb{P}[\forall i \in \llbracket \lfloor a\lambda^{-z} \rfloor, \lfloor b\lambda^{-z} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \xrightarrow{\lambda \rightarrow 0} 1$.
3. For $t \geq 1$, $\mathbb{P}[\forall i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \xrightarrow{\lambda \rightarrow 0} 1$.
4. For $t < 1$, $\mathbb{P}[\forall i \in \llbracket \lfloor a\mathbf{m}_\lambda \rfloor, \lfloor b\mathbf{m}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \xrightarrow{\lambda \rightarrow 0} 0$.
5. For $t > z_0 + \alpha$, $\mathbb{P}[\forall i \in \llbracket -\lfloor a\mathbf{m}_\lambda^\alpha \rfloor, \lfloor b\mathbf{m}_\lambda^\alpha \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \xrightarrow{\lambda \rightarrow 0} 1$.

Proof. To check Lemma 7.1, observe that, for $k_\lambda \xrightarrow{\lambda \rightarrow 0} \infty$,

$$\mathbb{P}[\forall i \in \llbracket -\lfloor ak_\lambda \rfloor, \lfloor bk_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \simeq (1 - e^{\mathbf{a}_\lambda t})^{(b-a)k_\lambda} \simeq e^{-(b-a)k_\lambda \lambda^t}. \quad (7.1)$$

In order to prove 1 and 2, use (7.1) with $k_\lambda = \lambda^{-z}$ and observe that

$$k_\lambda \lambda^t = \lambda^{-z} \lambda^t \xrightarrow{\lambda \rightarrow 0} \begin{cases} \infty & \text{if } t < z, \\ 0 & \text{if } t > z. \end{cases}$$

To prove 3, use (7.1) with $k_\lambda = \mathbf{n}_\lambda$ and observe that, if $t \geq 1$, $\mathbf{n}_\lambda \lambda^t \simeq \lambda^{t-1}/\mathbf{a}_\lambda$ tends to 0 when $\lambda \rightarrow 0$. In the same way, 4 can be proved using $k_\lambda = \mathbf{m}_\lambda$ and observing that, if $t < 1$, $\mathbf{m}_\lambda \lambda^t \simeq \lambda^{t-1}/\mathbf{a}_\lambda^2$ tends to ∞ when $\lambda \rightarrow 0$.

Finally, prove 5 with (7.1) and using $k_\lambda = \mathbf{m}_\lambda^\alpha$ and observing that $\mathbf{m}_\lambda^\alpha \lambda^t \simeq \frac{\alpha}{\mathbf{a}_\lambda} \lambda^{t-\alpha-(1-\alpha)z_0}$ tends to 0 when $\lambda \rightarrow 0$ as soon as $t - \alpha - (1-\alpha)z_0 > 0$ (in particular, for $t \geq z_0 + \alpha > \alpha + (1-\alpha)z_0$). \square

7.2 Height of the barrier

We describe here the time needed for a destroyed microscopic cluster to be regenerated. Assume that a match falls in the site 0 at some time $\mathbf{a}_\lambda t_1 \in (0, \mathbf{a}_\lambda z_0)$. As seen in **Micro**(∞, z_0) in Subsection 4.4, on a suitable event, the (λ, π) -FFP is well understood around 0 during $[\mathbf{a}_\lambda t_1, \mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^z)]$, for some $0 < z < z_0$ (it can be expressed using the sequence $(T_i^1)_{i \in \mathbb{Z}}$). We then denote by $\Theta_{t_1}^{\lambda, \pi}$ the delay needed for the destroyed cluster to be fully regenerated (after rescaling). We show that $\Theta_{t_1}^{\lambda, \pi} \simeq t_1$.

Lemma 7.2. *Consider two Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all this processes being independent. Let $0 < t_1 < z_0$. We call $(T_i^1)_{i \in \mathbb{Z}}$ the burning times of the propagation process ignited in 0 at time $\mathbf{a}_\lambda t_1$, recall Definition 4.6.*

Put, for all $t \geq 0$ and $i \in \mathbb{Z}$, $\zeta_t^{\lambda, \pi}(i) = \min(N_t^S(i), 1)$ and define

$$C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) = \llbracket i^g, i^d \rrbracket,$$

recall Definition 4.8.

*We define a process $(\zeta_{t_1, t}^{\lambda, \pi}(i))_{t \in [0, T], i \in \mathbb{Z}}$ in the following way (which is inspired by **Micro**(∞, z_0) in Subsection 4.4): we put, for all $i \in C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$*

$$\zeta_{t_1, t}^{\lambda, \pi}(i) = \min(N_{\mathbf{a}_\lambda t}^S(i), 1) \text{ for } t \in [0, t_1 + (T_i^1/\mathbf{a}_\lambda))$$

and

$$\zeta_{t_1, t}^{\lambda, \pi}(i) = 2 \begin{cases} \text{for } t \in [t_1 + (T_i^1/\mathbf{a}_\lambda), t_1 + (T_{i+1}^1/\mathbf{a}_\lambda)) & \text{if } i \geq 0, \\ \text{for } t \in [t_1 + (T_i^1/\mathbf{a}_\lambda), t_1 + (T_{i-1}^1/\mathbf{a}_\lambda)) & \text{if } i \leq 0 \end{cases}$$

and

$$\zeta_{t_1, t}^{\lambda, \pi}(i) = \begin{cases} \min(N_{\mathbf{a}_\lambda(t+t_1)}^S(i) - N_{\mathbf{a}_\lambda t_1 + T_{i+1}^1}^S(i), 1) & \text{for } t \in [t_1 + (T_{i+1}^1/\mathbf{a}_\lambda), T] \text{ if } i \geq 0, \\ \min(N_{\mathbf{a}_\lambda(t+t_1)}^S(i) - N_{\mathbf{a}_\lambda t_1 + T_{i-1}^1}^S(i), 1) & \text{for } t \in [t_1 + (T_{i-1}^1/\mathbf{a}_\lambda), T] \text{ if } i \leq 0. \end{cases}$$

For all $i \notin C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ and all $t \in [0, T]$, we put

$$\zeta_{t_1, t}^{\lambda, \pi}(i) = \min(N_{\mathbf{a}_\lambda t}^S(i), 1).$$

We finally define

$$\Theta_{t_1}^{\lambda, \pi} = \inf \left\{ t > t_1 : \forall i \in C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)), \zeta_{t_1, t}^{\lambda, \pi}(i) = 1 \right\}.$$

Then, for all $\delta > 0$, as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$, there holds

$$\lim_{\lambda, \pi} \mathbb{P} \left[|\Theta_{t_1}^{\lambda, \pi} - t_1| \geq \delta \right] = 0.$$

The process $(\zeta_{t_1, t}^{\lambda, \pi}(i))_{i \in \mathbb{Z}, t \geq 0}$ is closely related to the process observed in **Micro**(∞, z_0) in Subsection 4.4 (on a suitable event).

Proof. We divide the proof in two steps. We first define a simplest process with an instantaneous propagation: if a match falls in a cluster, it destroys instantaneously the entire connected component. The time needed for a microscopic cluster to become again occupied is almost t_1 . Secondly, we flank the killed cluster $C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ to estimate the time to become again occupied.

Step 1. Let $0 < \tau_1 < z_0$ be fixed. Put $\vartheta_t^\lambda(i) = \min(N_{\mathbf{a}_\lambda t}^S(i), 1)$ and $\vartheta_{\tau_1, t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_1+t)}^S(i) - N_{\mathbf{a}_\lambda \tau_1}^S(i), 1)$ for all $t > 0$ and $i \in \mathbb{Z}$. We define the time needed for the destroyed cluster to be fully regenerated

$$\Xi_{\tau_1}^\lambda = \inf \{t > 0 : \forall i \in C(\vartheta_{\tau_1, t}^\lambda, 0), \vartheta_{\tau_1, t}^\lambda(i) = 1\}.$$

Then for all $\delta > 0$,

$$\lim_{\lambda \rightarrow 0} \mathbb{P} [|\Xi_{\tau_1}^\lambda - \tau_1| \geq \delta] = 0.$$

Indeed, we write, for $h > 0$,

$$\begin{aligned} \mathbb{P} [\Xi_{\tau_1}^\lambda \leq h] &= \mathbb{P} [N_{\mathbf{a}_\lambda \tau_1}^S(0) = 0] + \sum_{k \geq 1} \sum_{j=0}^{k-1} \mathbb{P} [N_{\mathbf{a}_\lambda \tau_1}^S(j-k) = N_{\mathbf{a}_\lambda \tau_1}^S(j+1) = 0, \\ &\quad \forall i \in \llbracket j-k+1, j \rrbracket, N_{\mathbf{a}_\lambda \tau_1}^S(i) > 0, N_{\mathbf{a}_\lambda(\tau_1+h)}^S(i) > N_{\mathbf{a}_\lambda \tau_1}^S(i)], \end{aligned}$$

that is

$$\begin{aligned} \mathbb{P} [\Xi_{\tau_1}^\lambda \leq h] &= \lambda^{\tau_1} + \sum_{k \geq 1} \sum_{j=0}^{k-1} \lambda^{\tau_1} \times \lambda^{\tau_1} \times ((1 - \lambda^{\tau_1})(1 - \lambda^h))^k \\ &= \lambda^{\tau_1} + \lambda^{2\tau_1} \sum_{k \geq 1} k ((1 - \lambda^{\tau_1})(1 - \lambda^h))^k \\ &= \lambda^{\tau_1} + \frac{\lambda^{2\tau_1}}{(1 - (1 - \lambda^{\tau_1})(1 - \lambda^h))^2} (1 - \lambda^{\tau_1})(1 - \lambda^h) \\ &= \lambda^{\tau_1} + \frac{\lambda^{2\tau_1}}{(\lambda^{\tau_1} + \lambda^h - \lambda^{\tau_1+h})^2} (1 - \lambda^{\tau_1})(1 - \lambda^h). \end{aligned}$$

This quantity obviously tends to 1 as $\lambda \rightarrow 0$ if $h > \tau_1$ and to 0 if $h < \tau_1$.

Step 2. Let $z \in (t_1, z_0)$ and define $\Omega_{\lambda, \pi}^{P, z}(0, t_1)$, recall Definition 4.7. Set

$$\begin{aligned} \tilde{\Omega}_{\lambda, \pi}^{P, z}(0, t_1) &:= \Omega_{\lambda, \pi}^{P, z}(0, t_1) \cap \{\exists i_1 \in \llbracket 0, \lfloor \lambda^{-z} \rfloor \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^z)}^S(i_1) = 0\} \\ &\quad \cap \{\exists i_2 \in \llbracket -\lfloor \lambda^{-z} \rfloor, 0 \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^z)}^S(i_2) = 0\}. \end{aligned}$$

First, Lemma 4.4 together with Lemma 7.1-1 show that $\mathbb{P} [\tilde{\Omega}_{\lambda, \pi}^{P, z}(0, t_1)]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$ (because $t_1 + \kappa_{\lambda, \pi}^z < (z + t_1)/2 < z$ for (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$). Next, on $\tilde{\Omega}_{\lambda, \pi}^{P, z}(0, t_1)$, there holds that

$$C(\vartheta_{t_1 + \kappa_{\lambda, \pi}^z}^\lambda, 0) := \llbracket C^-, C^+ \rrbracket \subset \llbracket -\lfloor \lambda^{-z} \rfloor, \lfloor \lambda^{-z} \rfloor \rrbracket.$$

Since C^+ and C^- are vacant during $[\mathbf{a}_\lambda t_1, \mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^z)] \subset [0, \mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^z)]$, there holds that, as seen in **Micro**(∞, z_0) in Subsection 4.4,

$$C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) \subset C(\vartheta_{t_1 + \kappa_{\lambda, \pi}^z}^\lambda, 0) \subset \llbracket -\lfloor \lambda^{-z} \rfloor, \lfloor \lambda^{-z} \rfloor \rrbracket$$

and $\zeta_{\mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^z)}^{\lambda, \pi}(i) \leq 1$ for all $i \in \mathbb{Z}$. Besides, $C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ clearly contains $C(\vartheta_{t_1}^\lambda, 0)$, see Figure 5.

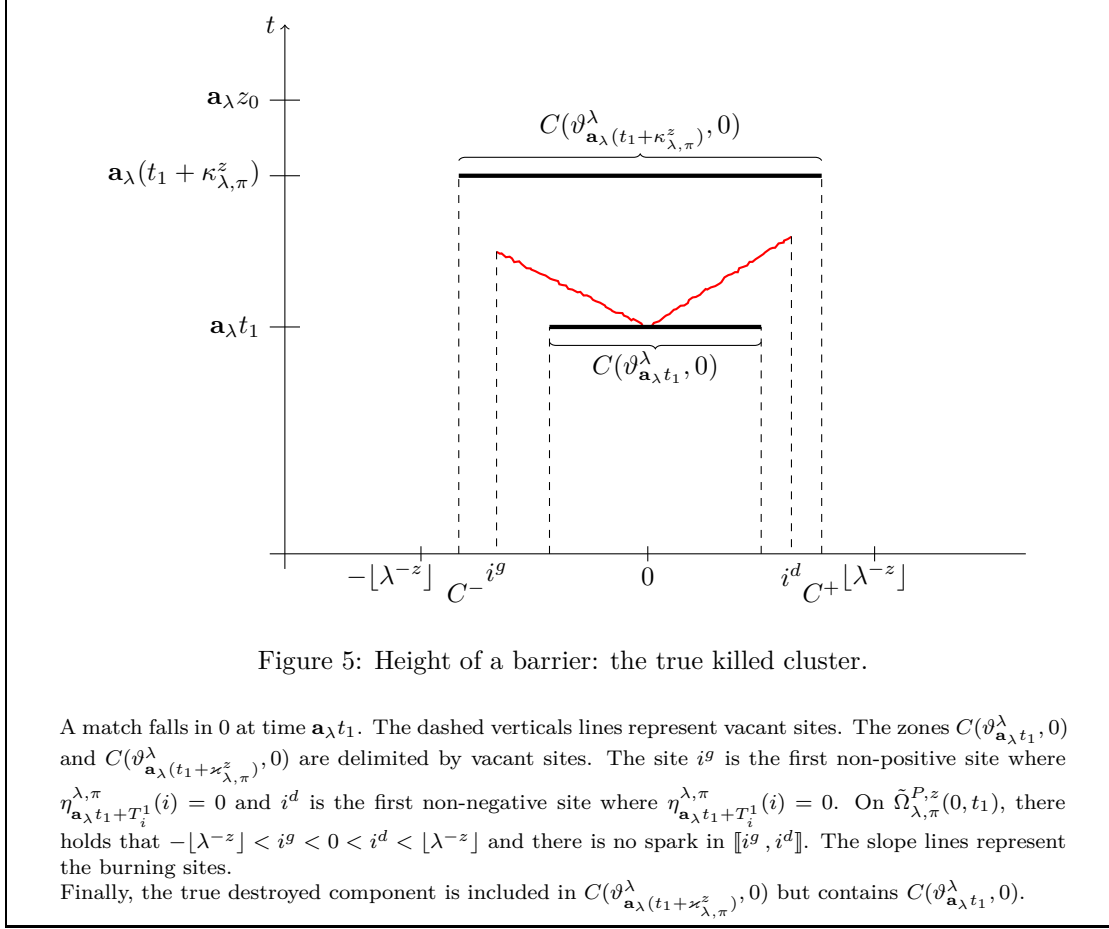
We trivially deduce that, conditionally on $\tilde{\Omega}_{\lambda, \pi}^{P, z}(0, t_1)$,

$$t_1 + \Xi_{t_1}^\lambda \leq t_1 + \Theta_{t_1}^{\lambda, \pi} \leq t_1 + \kappa_{\lambda, \pi}^z + \Xi_{t_1 + \kappa_{\lambda, \pi}^z}^\lambda.$$

Remark now that the function $t \mapsto t + \Xi_t^\lambda$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1 + \Theta_{t_1}^{\lambda, \pi} \xrightarrow{\lambda \rightarrow 0} 2t_1$$

in probability, whence for all $\delta > 0$ and all $\varepsilon > 0$, there holds that $\mathbb{P} [|\Theta_{t_1}^{\lambda, \pi} - t_1| \geq \delta] < \varepsilon$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$. \square



7.3 Proof of Theorem 6.2

Let us fix $z_0 \in [0, 1]$, $x_0 \in (-A, A)$, $t_0 > 0$ and $\varepsilon > 0$. The aim of this Section is to prove the

Lemma 7.3. *For all $\delta > 0$, there holds that*

$$\mathbb{P} \left[\delta(D_{t_0}^{\lambda, \pi, A}(x_0), D_{t_0}^A(x_0)) > \varepsilon \right] < \delta, \quad (7.2)$$

$$\mathbb{P} \left[\delta_T(D^{\lambda, \pi, A}(x_0), D^A(x_0)) > \varepsilon \right] < \delta, \quad (7.3)$$

for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

Clearly, (7.2) and (7.3) will imply the result. Let us first show that (7.2) (which holds for an arbitrary value of $t_0 \in (0, T)$) implies (7.3). Indeed, we have by construction for any $t \in [0, T]$, $\delta(D_t^{\lambda, \pi, A}(x_0), D_t^A(x_0)) < 4A$. Hence, by dominated convergence, (7.2) implies that $\mathbb{E} \left[\delta(D_t^{\lambda, \pi, A}(x_0), D_t^A(x_0)) \right] < \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$, whence again by dominated convergence, $\mathbb{E} \left[\delta_T(D^{\lambda, \pi, A}(x_0), D^A(x_0)) \right] < \delta$.

7.3.1 The coupling

We are going to construct a coupling between the (λ, π, A) -FFP (on the time interval $[0, a_\lambda T]$) and the LFFP(∞, z_0) (on $[0, T]$): we build the LFFP(∞, z_0) $(Y_t(x))_{t \in [0, T], x \in [-A, A]}$ from a Poisson measure π_M and we take for the matches for the discrete process the Poisson process

$$N_t^M(i) = \pi_M([i/\mathbf{n}_\lambda, (i+1)/\mathbf{n}_\lambda) \times [0, t/a_\lambda])$$

for all $i \in I_A^\lambda$ and $t \in [0, a_\lambda T]$.

We next introduce a family of i.i.d. Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective parameter 1 and π , independent of π_M .

The (λ, π, A) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in I_A^\lambda}$ is built from the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$, from the match processes $(N_t^M(i))_{t \geq 0, i \in \mathbb{Z}}$ and from the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$.

Observe that $(Y_t(x))_{t \in [0, T], x \in [-A, A]}$ is independent of $(N_t^S(i))_{t \in [0, \mathbf{a}_\lambda T], i \in I_A^\lambda}$ and $(N_t^P(i))_{t \in [0, \mathbf{a}_\lambda T], i \in I_A^\lambda}$.

When a match falls at some $x \in [-A, A]$ at some time $t \in [0, T]$ for the LFFP (∞, z_0) , it will fall at $\lfloor \mathbf{n}_\lambda x \rfloor$ at time $\mathbf{a}_\lambda t$ in the discrete process.

7.3.2 A sweet event

We call

$$n := \pi_M([0, T] \times [-A, A])$$

and we consider the marks $(T_q, X_q)_{q=1, \dots, n}$ of π_M ordered in such a way that $0 < T_1 < \dots < T_n < T$. We introduce

$$\mathcal{T}_M = \{T_1, \dots, T_n\} \text{ and } \mathcal{B}_M = \{X_1, \dots, X_n\}.$$

We also introduce

$$\mathcal{S}_M = \{2t : t \in \mathcal{T}_M, t < z_0\},$$

which has to be seen as the possible limit values of $t + \Theta_t^{\lambda, \pi} \simeq t + t$, recall Lemma 7.2.

For $\alpha > 0$, we consider the event

$$\Omega_M^0(\alpha) = \left\{ \min_{\substack{s \in \mathcal{T}_M \cup \mathcal{S}_M, \\ t \in \{0, z_0, t_0\}}} |t - s| > 2\alpha, \min_{\substack{x, y \in \mathcal{B}_M \cup \{x_0, -A, A\}, \\ x \neq y}} |x - y| > 2\alpha \right\},$$

which clearly satisfies $\lim_{\alpha \rightarrow 0} \mathbb{P}[\Omega_M^0(\alpha)] = 1$. For any given $\alpha \in (0, 1)$, on $\Omega_M^0(\alpha)$, there holds that for all $x, y \in \mathcal{B}_M \cup \{x_0\}$ with $x \neq y$, $(x)_\lambda^\alpha \cap (y)_\lambda^\alpha = \emptyset = (x)_\lambda \cap (y)_\lambda$.

We set

$$z_\alpha = (z_0 - \alpha) \vee (z_0/2).$$

For $q \in \{1, \dots, n\}$, using the seed processes family $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, we build, recall Definition 4.6, $(\zeta_t^{\lambda, \pi, q}(i))_{t \geq 0, i \in \mathbb{Z}}$ the propagation process ignited at (X_q, T_q) , $(i_t^{q, +})_{t \geq 0}$ and $(i_t^{q, -})_{t \geq 0}$ the corresponding right and left fronts, and $(T_i^q)_{i \in \mathbb{Z}}$ the associated burning times. We also define $\Omega_{\lambda, \pi}^{P, T, \alpha}(X_q, T_q)$ and $\Omega_{\lambda, \pi}^{P, z_\alpha}(X_q, T_q)$, recall Definition 4.7. If $z_0 \in (0, 1]$, we set

$$\Omega^{P, T}(\alpha, \lambda, \pi) = \bigcap_{q=1, \dots, n} (\Omega_{\lambda, \pi}^{P, T, \alpha}(X_q, T_q) \cap \Omega_{\lambda, \pi}^{P, z_\alpha}(X_q, T_q)).$$

If $z_0 = 0$, we simply set

$$\Omega^{P, T}(\alpha, \lambda, \pi) = \bigcap_{q=1, \dots, n} \Omega_{\lambda, \pi}^{P, T, \alpha}(X_q, T_q).$$

By Lemma 4.4 and since π_M is independent of $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, we deduce that $\mathbb{P}[\Omega^{P, T}(\alpha, \lambda, \pi)]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

Next we introduce the event $\Omega_1^S(\lambda, \pi)$ on which the following conditions hold: for all $q \in \{1, \dots, n\}$,

- if $T_q < z_\alpha$, there are $-\lfloor \lambda^{-z_\alpha} \rfloor < i_1^q < 0 < i_2^q < \lfloor \lambda^{-z_\alpha} \rfloor$ with $N_{\mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^{z_\alpha})}^S(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_1^q) = N_{\mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^{z_\alpha})}^S(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_2^q) = 0$;
- if $T_q > z_0 + \alpha$, for all $i \in (X_q)_\lambda^\alpha$, $N_{\mathbf{a}_\lambda T_q}^S(i) > 0$.

Since $\kappa_{\lambda, \pi}^{z_\alpha}$ can be made arbitrarily small in the regime $\mathcal{R}(\infty, z_0)$, Lemma 7.1 then show that $\mathbb{P}[\Omega_1^S(\lambda, \pi)]$ tends to 1 when $\lambda \rightarrow 0$ and $\rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

We also consider the event $\Omega_2^S(\lambda)$ on which the following conditions holds

- if $t_0 < 1$, there are $\lfloor \mathbf{n}_\lambda x_0 \rfloor - \mathbf{m}_\lambda < i_1^0 < \lfloor \mathbf{n}_\lambda x_0 \rfloor < i_2^0 < \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda$ with $N_{\mathbf{a}_\lambda t_0}^S(i_1) = N_{\mathbf{a}_\lambda t_0}^S(i_2) = 0$;
- for all $i \in \llbracket -A_\lambda, A_\lambda \rrbracket$, $N_{\mathbf{a}_\lambda}^S(i) > 0$.

Lemma 7.1 together with space/time stationarity implies that $\lim_{\lambda \rightarrow 0} \mathbb{P}[\Omega_2^S(\lambda)] = 1$.

We also need $\Omega_3^{S,P}(\gamma, \lambda, \pi)$, defined for $\gamma > 0$ as follows: for all $q = 1, \dots, n$ with $T_q < z_0$, there holds that $|\Theta_{T_q}^{\lambda, \pi, q} - T_q| < \gamma$. Here $\Theta_{T_q}^{\lambda, \pi, q}$ is defined as in Lemma 7.2 with the seed processes family $(N_t^{S,q}(i))_{t \geq 0, i \in \mathbb{Z}} = (N_t^S(i + \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^{P,q}(i))_{t \geq 0, i \in \mathbb{Z}} = (N_t^P(i + \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0, i \in \mathbb{Z}}$. Lemma 7.2 directly implies that for any $\gamma > 0$, $\mathbb{P}[\Omega_3^{S,P}(\gamma, \lambda, \pi)]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(\infty, z_0)$.

We finally introduce the event

$$\Omega(\alpha, \gamma, \lambda, \pi) = \Omega_M^0(\alpha) \cap \Omega^{P,T}(\alpha, \lambda, \pi) \cap \Omega_1^S(\lambda, \pi) \cap \Omega_2^S(\lambda) \cap \Omega_3^{S,P}(\gamma, \lambda, \pi).$$

We have shown that for any $\delta > 0$, there exists $\alpha \in (0, 1)$ such that for any $\gamma > 0$, there holds $\mathbb{P}[\Omega(\alpha, \gamma, \lambda, \pi)] > 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$.

7.3.3 Heart of the proof

The next Lemma is the key of the proof: it guarantees that each fire have a local effect. It will be repeatedly used in Lemmas 7.5 and 7.6.

Lemma 7.4. *On $\Omega(\alpha, \gamma, \lambda, \pi)$, the match falling on $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$, for some $q \in \{1, \dots, n\}$, does not affect the zone outside $(X_q)_\lambda^\alpha$ during $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda T]$.*

Consequently, on $\Omega(\alpha, \gamma, \lambda, \pi)$, for all $i \in I_A^\lambda \setminus \cup_{q=1, \dots, n} (X_q)_\lambda^\alpha$ and all $t \in [0, T]$, there holds that

$$\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(i) = \min(N_{\mathbf{a}_\lambda t}^S(i), 1).$$

Proof. As be seen in **Macro** (∞, z_0) in Subsection 4.4, on $\Omega_{\lambda, \pi}^{P,T,\alpha}(X_q, T_q) \subset \Omega(\alpha, \gamma, \lambda, \pi)$, there holds that

$$X_q - \frac{\mathbf{m}_\lambda^\alpha}{\mathbf{n}_\lambda} \leq \frac{\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda T}^{q,-}}{\mathbf{n}_\lambda} \leq X_q \leq \frac{\lfloor \mathbf{n}_\lambda X_q \rfloor + 1 + i_{\mathbf{a}_\lambda T}^{q,+}}{\mathbf{n}_\lambda} \leq X_q + \frac{\mathbf{m}_\lambda^\alpha}{\mathbf{n}_\lambda}$$

with $\mathbf{m}_\lambda^\alpha / \mathbf{n}_\lambda \leq \alpha$. Hence, each fire has only a local effect and does not affect the zone outside $(X_q)_\lambda^\alpha$. \square

We now turn to fires of the second kind.

Lemma 7.5. *Let $q \in \{1, \dots, n\}$ such that $T_q > z_0 + \alpha$. On $\Omega(\alpha, \gamma, \lambda, \pi)$, for all $t \in [\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda T]$, there holds that*

$$\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(t-T_q)}^{q,+}) = 2 = \eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(t-T_q)}^{q,-}).$$

Proof. At time $\mathbf{a}_\lambda T_q -$, at least one seed has fallen on each site of $(X_q)_\lambda^\alpha$, thanks to $\Omega_1^S(\lambda, \pi)$. Thus, the zone $(X_q)_\lambda^\alpha$ is completely filled at time $\mathbf{a}_\lambda T_q -$, thanks to Lemma 7.4 (no fire can affect this zone during $[0, \mathbf{a}_\lambda T_q)$). The conclusion is then straightforward, since on $\Omega_{\lambda, \pi}^{P,T}(X_q, T_q)$ there holds that $i^{q,+} \leq \mathbf{m}_\lambda^\alpha / \mathbf{n}_\lambda$ and $i^{q,-} \leq \mathbf{m}_\lambda^\alpha / \mathbf{n}_\lambda$ (as seen in **Macro** (∞, z_0) in Subsection 4.4) and since no match falling outside $(X_q)_\lambda^\alpha$ can affect this zone. \square

Finally, we treat the case of the fires of the first kind.

Lemma 7.6. *Let $q \in \{1, \dots, n\}$ such that $T_q < z_0 - \alpha$. On $\Omega(\alpha, \gamma, \lambda, \pi)$, there holds that*

$$(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(i))_{t \in [0, T], i \in (X_q)_\lambda^\alpha} = (\zeta_{T_q, t}^{\lambda, \pi, q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \in [0, T], i \in (X_q)_\lambda^\alpha},$$

where the last process is defined as in Lemma 7.2, using the seed processes family $(N_t^{S,q}(i))_{t \geq 0, i \in \mathbb{Z}} = (N_t^S(i + \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^{P,q}(i))_{t \geq 0, i \in \mathbb{Z}} = (N_t^P(i + \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0, i \in \mathbb{Z}}$.

Consequently, on $\Omega(\alpha, \gamma, \lambda, \pi)$, for some $\gamma \in (0, \alpha)$,

(a) if $t \in [T_q + \alpha, 2T_q - \alpha]$, then there exists $i \in (X_q)_\lambda^\alpha$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 0$,

(b) if $t \geq (2T_q + \alpha) \vee 1$, then $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 1$ for all $i \in (X_q)_\lambda^\alpha$.

Proof. First observe that the process $(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i))_{t \in [0, T], i \in \llbracket -\mathbf{m}_\lambda^\alpha, \mathbf{m}_\lambda^\alpha \rrbracket}$ and the process $(\zeta_{T_q, t}^{\lambda, \pi, q}(i))_{t \in [0, T], i \in \llbracket -\mathbf{m}_\lambda^\alpha, \mathbf{m}_\lambda^\alpha \rrbracket}$ evolve according to the same seed processes family and to the same propagation processes family.

Lemma 7.4 implies that, for all $i \in (X_q)_\lambda^\alpha$ and all $t \in [0, T_q]$,

$$\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = \min(N_{\mathbf{a}_\lambda t}^S(i), 1),$$

because no match falls in $(X_q)_\lambda^\alpha$ during $[0, \mathbf{a}_\lambda T_q]$. This in particular implies that, for all $i \in (X_q)_\lambda^\alpha$ and all $t \in [0, T_q]$,

$$\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = \zeta_{T_q, t}^{\lambda, \pi, q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor).$$

On $\Omega_{\lambda, \pi}^{P, z_\alpha}(X_q, T_q) \cap \Omega_1^S(\lambda, \pi)$, as seen in **Micro**(∞, z_0) in Subsection 4.4, since the two processes are building using the same seed processes family and the same propagation processes family, there also holds true that for all $i \in (X_q)_\lambda^\alpha$ and all $t \in [T_q, T_q + \kappa_{\lambda, \pi}^{z_\alpha}]$,

$$\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = \zeta_{T_q, t}^{\lambda, \pi, q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor).$$

Finally, since there is no more burning tree in $(X_q)_\lambda^\alpha$ at time $\mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^{z_\alpha})$ and since seeds fall according to the same processes, we deduce that, thanks again to Lemma 7.4, the two processes remain equal during $(T_q + \kappa_{\lambda, \pi}^{z_\alpha}, T]$.

All this implies that

$$(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi, A}(i))_{t \in [0, T], i \in (X_q)_\lambda^\alpha} = (\zeta_{T_q, t}^{\lambda, \pi, q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \in [0, T], i \in (X_q)_\lambda^\alpha}. \quad (7.4)$$

Consider now the zone destroyed by the match falling on $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$

$$C^P := C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_q, T_q)).$$

As seen in **Micro**(∞, z_0) in Subsection 4.4, $C^P \subset \llbracket -\lfloor \lambda^{-z_\alpha} \rfloor, \lfloor \lambda^{-z_\alpha} \rfloor \rrbracket$ because there are $i_1 \in \llbracket -\lfloor \lambda^{-z_\alpha} \rfloor, 0 \rrbracket$ and $i_2 \in \llbracket 0, \lfloor \lambda^{-z_\alpha} \rfloor \rrbracket$ which are vacant until $\mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^{z_\alpha})$, thanks to $\Omega_1^S(\lambda, \pi)$.

From (7.4) and since no match falling outside $(X_q)_\lambda^\alpha$ can affect this zone, it follows that

$$\Theta_{T_q}^{\lambda, \pi, q} = \inf \left\{ t > T_q : \forall i \in C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_q, T_q)), \eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 1 \right\}.$$

Hence, the zone C^P is not completely occupied during $(\mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^{z_\alpha}), \mathbf{a}_\lambda(T_q + \Theta_{T_q}^{\lambda, \pi, q}))$ but is completely filled at time $\mathbf{a}_\lambda(T_q + \Theta_{T_q}^{\lambda, \pi, q})$.

Using $\Omega_3^{S, P}(\gamma, \lambda, \pi) \cap \Omega_M^0(\alpha)$ and since $\gamma \in (0, \alpha)$, we deduce that,

$$T_q + \alpha < 2T_q - \alpha \leq 2T_q - \gamma \leq T_q + \Theta_{T_q}^{\lambda, \pi, q} \leq 2T_q + \gamma \leq 2T_q + \alpha.$$

We now conclude.

(a) If $t \in [T_q + \alpha, 2T_q - \alpha]$, then the zone C^P is not completely occupied at time t . Hence, there exists $i \in C^P \subset (X_q)_\lambda^\alpha$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 0$.

(b) If $t \geq (2T_q + \alpha) \vee 1$, then C^P is completely filled at time t because $t \geq T_q + \alpha$.

Consider now $i \in (X_q)_\lambda^\alpha \setminus C^P$. Then i has not been killed by the fire starting at $\lfloor \mathbf{n}_\lambda X_q \rfloor$. Thus i cannot have been killed during $[0, \mathbf{a}_\lambda t] \supset [0, \mathbf{a}_\lambda]$, thanks to Lemma 7.4. We conclude using that $t \geq 1$, so that on $\Omega_1^S(\lambda)$, i is occupied at time $\mathbf{a}_\lambda t$. \square

7.3.4 Conclusion

First, the case $t_0 < 1$ is simple.

Lemma 7.7. *For $t_0 < 1$, on $\Omega(\alpha, \gamma, \lambda, \pi)$, there holds that*

$$\delta(D_{t_0}^{\lambda, \pi, A}(x_0), D_{t_0}^A(x_0)) < \frac{2\mathbf{m}_\lambda}{\mathbf{n}_\lambda}.$$

Proof. Thanks to $\Omega_2^S(\lambda)$, there are $i_1^0 \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x_0 \rfloor \rrbracket$ and $i_2^0 \in \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda \rrbracket$ such that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi, A}(i_1) = \eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi, A}(i_2) = 0$. Thus, $C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda \rrbracket$ whence $D_{t_0}^{\lambda, \pi, A}(x_0) \subset [x_0 - \mathbf{m}_\lambda/\mathbf{n}_\lambda, x_0 + \mathbf{m}_\lambda/\mathbf{n}_\lambda]$. Since $D_{t_0}^A(x_0) = \{x_0\}$, we deduce that

$$\delta(D_{t_0}^{\lambda, \pi, A}(x_0), D_{t_0}^A(x_0)) \leq \frac{2\mathbf{m}_\lambda}{\mathbf{n}_\lambda}. \quad \square$$

We now turn to the case $t_0 \geq 1$.

Lemma 7.8. *For $t_0 \geq 1$, on $\Omega(\alpha, \gamma, \lambda, \pi)$ for some $0 < \gamma < \alpha$ and for all (λ, π) sufficiently close to the regime $\mathcal{R}(\infty, z_0)$ in such a way that $\kappa_{\lambda, \pi}^{z_\alpha} \leq \alpha$ and $\lfloor z^{-\alpha} \rfloor \leq \mathbf{m}_\lambda^\alpha$, there holds that*

$$\delta(D_{t_0}^{\lambda, \pi, A}(x_0), D_{t_0}^A(x_0)) < \frac{2\mathbf{m}_\lambda^\alpha}{\mathbf{n}_\lambda}.$$

Proof. Clearly, since $t_0 \geq 1$, $D_{t_0}^A(x_0) = [a, b]$ for some $a, b \in \mathcal{B}_M \cup \{-A, A\}$. Assume $-A < a < b < A$, the other cases being treated similarly. In the limit process, we then have $Y_{t_0}(a) > 0$, $Y_{t_0}(b) > 0$ and $Y_{t_0}(x) = 0$ for all $x \in (a, b)$. We will prove separately that

1. there are $i \in (a)_\lambda^\alpha$ and $j \in (b)_\lambda^\alpha$ such that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi, A}(i) = 0$ or 2 and $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi, A}(j) = 0$ or 2;
2. for all $x \in \mathcal{B}_M \cap (a, b)$, for all $i \in (x)_\lambda^\alpha$, $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi, A}(i) = 1$;
3. for all $i \in \llbracket \lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda^\alpha + 1, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda^\alpha - 1 \rrbracket \setminus \cup_{x \in \mathcal{B}_M \cap (a, b)} (x)_\lambda^\alpha$, we have $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi, A}(i) = 1$.

Points 1., 2. and 3. imply that,

$$\llbracket \lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda^\alpha + 1, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda^\alpha - 1 \rrbracket \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset \llbracket \lfloor \mathbf{n}_\lambda a \rfloor - \mathbf{m}_\lambda^\alpha - 1, \lfloor \mathbf{n}_\lambda b \rfloor + \mathbf{m}_\lambda^\alpha + 1 \rrbracket$$

and thus $[a + \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda, b - \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda] \subset D_{t_0}^{\lambda, \pi, A}(x_0) \subset [a - \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda, b + \mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda]$, whence,

$$\delta(D_{t_0}^A(x_0), D_{t_0}^{\lambda, \pi, A}(x_0)) \leq 2\mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda.$$

We prove 1. Let $k \in \{1, \dots, n\}$ such that $a = X_k$. There are two cases.

Case 1. If $Y_{t_0}(X_k) = 1$ in the limit process, then $t_0 \geq T_k \geq z_0$ whence $t_0 \geq T_k \geq z_0 + 2\alpha$ due to $\Omega_M^0(\alpha)$. We then use Lemma 7.5 and conclude that there is a burning tree in $(a)_\lambda^\alpha$ at time $\mathbf{a}_\lambda t_0$.

Case 2. If $Y_{t_0}(a) \in (0, 1)$ in the limit process, then $T_k \leq z_0 \leq 1 \leq t_0 \leq 2T_k$ whence $T_k + 4\alpha \leq z_0 + 2\alpha \leq t_0 + 2\alpha \leq 2T_k$, due to $\Omega_M^0(\alpha)$. We conclude using Lemma 7.6-(a) that there is a vacant site in $(a)_\lambda^\alpha$ at time $\mathbf{a}_\lambda t_0$.

Similar considerations hold for b .

We prove 2. Let $x \in \mathcal{B}_M \cap (a, b)$ and let $k \in \{1, \dots, n\}$ such that $x = X_k$.

Case 1. If $T_k > t_0$, then no fire has fallen in $(X_k)_\lambda^\alpha$ during $[0, \mathbf{a}_\lambda t_0]$. Using $\Omega_1^S(\lambda, \pi)$ and Lemma 7.4, we conclude that $(X_k)_\lambda^\alpha$ is completely occupied at time $\mathbf{a}_\lambda t_0$ (because no fire can affect this zone).

Case 2. If $T_k \leq t_0$, since in the limit process $Y_{t_0}(X_k) = 0$, necessarily $T_k \leq z_0 \leq t_0$ and $2T_k \leq t_0$ whence $T_k \leq z_0 - 2\alpha$ and $2T_k \leq t_0 - 2\alpha$ due to $\Omega_M(\alpha)$. Lemma 7.6-(b) concludes this case since $t_0 \geq (2T_k + \alpha) \vee 1$.

We prove 3. Let $i \in \llbracket \lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda^\alpha + 1, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda^\alpha - 1 \rrbracket \setminus \cup_{j=1, \dots, n} (X_j)_\lambda^\alpha$, using Lemma 7.4 and $\Omega_2^S(\lambda)$, we immediately conclude that i is occupied at time $\mathbf{a}_\lambda t_0$. \square

We now can conclude.

Proof of Lemma 7.3. Let $\delta > 0$ be fixed. We first consider $\alpha_0 \in (0, \varepsilon/2)$, $\gamma_0 \in (0, \alpha_0)$, $\lambda_0 \in (0, 1]$, $\epsilon_0 > 0$ and $K_0 \geq 1$ such that for all $\lambda \in (0, \lambda_0)$, all $\pi \geq 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \geq K_0$ and $\left| \frac{\log(\pi)}{\log(1/\lambda)} - z_0 \right| < \epsilon_0$, we have

$$\mathbb{P}[\Omega(\alpha_0, \gamma_0, \lambda, \pi)] > 1 - \delta.$$

Then we consider $\lambda_1 \in (0, \lambda_0)$, $K_1 > K_0$ and $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\lambda \in (0, \lambda_1)$ and all $\pi \geq 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} \geq K_1$ and $\left| \frac{\log(\pi)}{\log(1/\lambda)} - z_0 \right| < \epsilon_1$, we have

- $2\mathbf{m}_\lambda/\mathbf{n}_\lambda < \varepsilon$,
- $\kappa_{\lambda, \pi}^{z_\alpha} < \alpha$,
- $2\lambda^{-z_\alpha}/\mathbf{n}_\lambda < 2\mathbf{m}_\lambda^\alpha/\mathbf{n}_\lambda < \varepsilon$.

For all $\lambda \in (0, \lambda_1)$, all $\pi \geq 1$ in such a way that $\frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} > K_1$ and $\left| \frac{\log(\pi)}{\log(1/\lambda)} - z_0 \right| < \epsilon_1$, Lemma 7.7 implies that, if $t_0 < 1$,

$$\mathbb{P}\left[\delta(D_{t_0}^A(x_0), D_{t_0}^{\lambda, \pi, A}(x_0)) > \varepsilon\right] \leq \mathbb{P}\left[\delta(D_{t_0}^A(x_0), D_{t_0}^{\lambda, \pi, A}(x_0)) > \frac{2\mathbf{m}_\lambda}{\mathbf{n}_\lambda}\right] \leq \mathbb{P}[\Omega(\alpha_0, \gamma_0, \lambda, \pi)^c] < \delta$$

while, if $t_0 \geq 1$, Lemma 7.8 implies that, (since $\alpha \geq \gamma$ and $\alpha \geq \kappa_{\lambda, \pi}^{z_\alpha}$)

$$\mathbb{P}\left[\delta(D_{t_0}^A(x_0), D_{t_0}^{\lambda, \pi, A}(x_0)) > \varepsilon\right] \leq \mathbb{P}\left[\delta(D_{t_0}^A(x_0), D_{t_0}^{\lambda, \pi, A}(x_0)) > \frac{2\mathbf{m}_\lambda^{\alpha_0}}{\mathbf{n}_\lambda}\right] \leq \mathbb{P}[\Omega(\alpha_0, \gamma_0, \lambda, \pi)^c] < \delta.$$

This concludes the proof. \square

8 Convergence in the regime $\mathcal{R}(p)$, for some $p > 0$

The aim of this section is to prove Theorem 6.1 for $p > 0$ and this will conclude the proof of Theorem 2.5 for $p > 0$.

In the whole section, we fix the parameters $A > 0$, $T > 2$ and $p > 0$. We omit the subscript/superscript A in the whole proof.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$, $\mathbf{m}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda^2) \rfloor$, $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$. We set as usual $A_\lambda = \lfloor \mathbf{n}_\lambda A \rfloor$ and $I_A^\lambda = \llbracket -A_\lambda, A_\lambda \rrbracket$. For $i \in \mathbb{Z}$, we set $i_\lambda = \lfloor i/\mathbf{n}_\lambda, (i+1)/\mathbf{n}_\lambda \rfloor$. For $[a, b]$ an interval of $[-A, A]$ and $\lambda \in (0, 1)$, we introduce, assuming that $-A < a < b < A$,

$$\begin{aligned} [a, b]_\lambda &= \llbracket \lfloor \mathbf{n}_\lambda a + \mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{n}_\lambda b - \mathbf{m}_\lambda \rfloor - 1 \rrbracket \subset \mathbb{Z}, \\ [-A, b]_\lambda &= \llbracket -A_\lambda, \lfloor \mathbf{n}_\lambda b - \mathbf{m}_\lambda \rfloor - 1 \rrbracket \subset \mathbb{Z}, \\ [a, A]_\lambda &= \llbracket \lfloor \mathbf{n}_\lambda a + \mathbf{m}_\lambda \rfloor + 1, A_\lambda \rrbracket \subset \mathbb{Z}. \end{aligned}$$

For $\lambda \in (0, 1)$ and $\pi \geq 1$, we recall that

$$\kappa_{\lambda, \pi}^0 = \frac{\mathbf{m}_\lambda}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$$

and introduce

$$\mathbf{k}_{\lambda, \pi} = \lfloor \mathbf{a}_\lambda \pi (\varepsilon_\lambda + \mathbf{v}_{\lambda, \pi}) \rfloor, \quad (8.1)$$

$$\mathbf{v}_{\lambda, \pi} = \kappa_{\lambda, \pi}^0 + \mathbf{v}_{\lambda, \pi}, \quad (8.2)$$

$$\mathbf{e}_{\lambda, \pi} = \varepsilon_\lambda + \mathbf{v}_{\lambda, \pi}, \quad (8.3)$$

where $\mathbf{v}_{\lambda, \pi} = \left(\frac{T}{p} \vee 2A \right) \left| \frac{\mathbf{n}_\lambda}{\mathbf{a}_\lambda \pi} - p \right|$. Observe that $\mathbf{k}_{\lambda, \pi}/\mathbf{n}_\lambda$, $\mathbf{v}_{\lambda, \pi}$ and $\mathbf{e}_{\lambda, \pi}$ tend to 0 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

For $x \in (-A, A)$, $\lambda \in (0, 1)$ and $\pi \geq 1$, we introduce

$$(x)_\lambda = \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda \rrbracket \subset \mathbb{Z}, \quad (8.4)$$

$$\langle x \rangle_{\lambda, \pi} = \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{k}_{\lambda, \pi} \rrbracket \subset \mathbb{Z}, \quad (8.5)$$

$$[x]_{\lambda, \pi} = \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi} \rrbracket \subset \mathbb{Z}. \quad (8.6)$$

8.1 Occupation of vacant zone

We start with some easy estimates.

Lemma 8.1. *Consider a family of i.i.d. Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$. Let $a < b$.*

1. *For $t < 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket \lfloor a\mathbf{m}_\lambda \rfloor, \lfloor b\mathbf{m}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$;*
2. *For $t \geq 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket \lfloor a\mathbf{m}_\lambda \rfloor, \lfloor b\mathbf{m}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
3. *For $t < 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$;*
4. *For $t \geq 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
5. *For $t > 0$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\exists i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
6. *For $t > 0$ and $\delta > 0$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket -\lfloor \lambda^{-(t+\delta)} \rfloor, \lfloor \lambda^{-(t+\delta)} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$;*
7. *For $t > 0$ and $\delta > 0$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket -\lfloor \lambda^{-(t-\delta)} \rfloor, \lfloor \lambda^{-(t-\delta)} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
8. *For $t < 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket \lfloor a\mathbf{k}_{\lambda, \pi} \rfloor, \lfloor b\mathbf{k}_{\lambda, \pi} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$ (when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$);*
9. *For $t \geq 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P} [\forall i \in \llbracket \lfloor a\mathbf{k}_{\lambda, \pi} \rfloor, \lfloor b\mathbf{k}_{\lambda, \pi} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$ (when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$).*

Proof. This lemma is closely related to Lemma 7.1. For $r_\lambda \xrightarrow{\lambda \rightarrow 0} \infty$, we have

$$\mathbb{P} [\forall i \in \llbracket -\lfloor ar_\lambda \rfloor, \lfloor br_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \simeq (1 - e^{-\mathbf{a}_\lambda t})^{(b-a)r_\lambda} \simeq e^{-(b-a)r_\lambda t}.$$

Observe now that

$$\mathbf{m}_\lambda \lambda^t \simeq \frac{\lambda^{t-1}}{\mathbf{a}_\lambda^2} \xrightarrow{\lambda \rightarrow 0} \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

from which points 1 and 2 follow, that

$$\mathbf{n}_\lambda \lambda^t \simeq \frac{\lambda^{t-1}}{\mathbf{a}_\lambda} \xrightarrow{\lambda \rightarrow 0} \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

which implies points 3 and 4. For the point 5, it suffices to note that, for any $i \in \mathbb{Z}$,

$$\mathbb{P} [N_{\mathbf{a}_\lambda t}^S(i) = 0] = e^{-\mathbf{a}_\lambda t}.$$

Hence

$$\mathbb{P} [\exists i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \simeq 1 - e^{-\mathbf{a}_\lambda \mathbf{n}_\lambda t (b-a)} \xrightarrow{\lambda \rightarrow 0} 1.$$

For $t > 0$ and $\delta > 0$, we have

$$\mathbb{P} [\forall i \in \llbracket -\lfloor \lambda^{-(t+\delta)} \rfloor, \lfloor \lambda^{-(t+\delta)} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \simeq e^{-2\lambda^{-\delta}} \xrightarrow{\lambda \rightarrow 0} 0,$$

which prove point 6, whence

$$\mathbb{P} [\forall i \in \llbracket -\lfloor \lambda^{-(t-\delta)} \rfloor, \lfloor \lambda^{-(t-\delta)} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] \simeq e^{-2\lambda^\delta} \xrightarrow{\lambda \rightarrow 0} 1$$

which is Point 7.

For the two last statement, as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, we have, observing that $\mathbf{v}_{\lambda, \pi} \rightarrow 0$,

$$\mathbf{k}_{\lambda, \pi} \lambda^t \simeq \mathbf{a}_\lambda \pi \lambda^t (\varepsilon_\lambda + \mathbf{v}_{\lambda, \pi}) \simeq \frac{\mathbf{n}_\lambda \lambda^t}{p} (\varepsilon_\lambda + \mathbf{v}_{\lambda, \pi}) \simeq \frac{\lambda^{t-1}}{\mathbf{a}_\lambda p} (1/\mathbf{a}_\lambda^3 + \mathbf{v}_{\lambda, \pi}) \xrightarrow{\lambda, \pi} \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1. \end{cases} \quad \square$$

8.2 Height of the barrier

We describe here the time needed for a destroyed microscopic cluster to be regenerated. Roughly, we assume that the zone $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ around 0 has been made vacant at some time $\mathbf{a}_\lambda t_0$. Then we consider the situation where a match falls on 0 at some time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + 1))$ and we compute the delay needed for the destroyed cluster to be fully regenerated. We have to distinguish two cases.

- a) We first consider the case where a match falls on 0 at time $\mathbf{a}_\lambda t_1 \in (0, \mathbf{a}_\lambda)$. This case is closely related to Lemma 7.2.
- b) We then consider the case where a fire propagates through $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda t_0$ and a match falls on 0 at time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + 1))$. This case is a little bit different but is proved in the same way as the previous case.

Lemma 8.2. *Consider two Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all this processes being independent. Consider also $\mathcal{M} := (i_0; t_0, t_1) \in \mathbb{Z} \times (\mathbb{R}_+)^2$ with $|i_0| \in \llbracket \mathbf{m}_\lambda, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi} \rrbracket$, $t_0 \in \{0\} \cup (1, \infty)$ and $t_1 \in (t_0, t_0 + 1)$. For $i \in \mathbb{Z}$ and $t \geq 0$, we consider the process*

$$\begin{aligned} \zeta_t^{\lambda, \pi, \mathcal{M}}(i) &= (1 + \mathbf{1}_{\{t \geq \mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda, \pi}), i = i_0\}}) \times \mathbf{1}_{\{t_0 > 1\}} \\ &+ \mathbf{1}_{\{t \geq \mathbf{a}_\lambda t_1, i = 0, \zeta_{\mathbf{a}_\lambda t_1 -}^{\lambda, \pi, \mathcal{M}}(0) = 1\}} + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 0\}} dN_s^S(i) \\ &+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i+1) = 2, \zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 1\}} dN_s^P(i+1) \\ &+ \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i-1) = 2, \zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 1\}} dN_s^P(i-1) \\ &- 2 \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 2\}} dN_s^P(i). \end{aligned}$$

Using the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, consider the burning times $(T_i^1)_{i \in \mathbb{Z}}$ of the propagation process ignited at $(0, t_1)$, recall Definition 4.6, and define the destroyed cluster due to the match falling in 0 at time $\mathbf{a}_\lambda t_1$, recall (4.12),

$$C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) := \llbracket i^g, i^d \rrbracket.$$

We finally define the time needed for $C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ to become again occupied

$$\Theta_{\mathcal{M}}^{\lambda, \pi} := \inf \left\{ t > t_1 : \forall i \in C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)), \zeta_{\mathbf{a}_\lambda t}^{\lambda, \pi, \mathcal{M}}(i) = 1 \right\}.$$

For all $\delta > 0$, there holds that,

$$\lim_{\lambda, \pi} \mathbb{P} \left[\left| \Theta_{\mathcal{M}}^{\lambda, \pi} - (t_1 - t_0) \right| \geq \delta \right] = 0$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

Let us explain the behaviour of the process $(\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}$. If $t_0 = 0$, then the process starts from a vacant initial situation and a match falls on 0 at time $\mathbf{a}_\lambda t_1$. It does not depend on i_0 and since $0 < t_1 < 1$, the zone $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ is not completely filled at time $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^0)$, see Lemma 8.1-1 (and because $\kappa_{\lambda, \pi}^0 \rightarrow 0$). The process is then governed by the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ and the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ with the same rules as the (λ, π) -FFP. As seen in **Micro**(p) in Subsection 4.4, the fire is extinguished at time $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda, \pi}^0)$.

If $t_0 > 1$, then the process starts at time 0 from an occupied initial situation, nothing happens until a match falls on i_0 at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda, \pi})$. Two fires start: one goes to the left and one goes to the right. Thus, on $\Omega_{\lambda, \pi}^{P, T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda, \pi})$, recall Definition 4.7, and since

$$\lfloor \mathbf{a}_\lambda \pi(3\mathbf{v}_{\lambda, \pi} - \varepsilon_\lambda) \rfloor \geq 2\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi},$$

recall (8.1) and (8.2), each site of $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ burns and extinguishes before $\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})$, recall Lemma 4.2. Hence, the zone $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ is not completely filled when the match falls on 0 at time $\mathbf{a}_\lambda t_1$, see Lemma 8.1-1 and because $\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi}) < \mathbf{a}_\lambda t_1 < \mathbf{a}_\lambda(t_0 + 1)$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

Proof. The proof is in the same spirit as the proof of Lemma 7.2. We first define the simplest process with an instantaneous propagation: if a match falls in a cluster, it destroys instantaneously the entire connected component. Secondly, we flank the killed cluster $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ to estimate the time needed to become again occupied, see Figure 6.

Step 1. Let $\tau_0 < \tau_1 < \tau_0 + 1$ be fixed. Put $\vartheta_{\tau_0,t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_0+t)}^S(i) - N_{\mathbf{a}_\lambda\tau_0}^S(i), 1)$ and $\vartheta_{\tau_1,t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_1+t)}^S(i) - N_{\mathbf{a}_\lambda\tau_1}^S(i), 1)$ for all $t > 0$ and all $i \in \mathbb{Z}$. We define the time needed for the destroyed cluster to be fully regenerated

$$\Xi_{\tau_0,\tau_1}^\lambda = \inf \{t > 0 : \forall i \in C(\vartheta_{\tau_0,\tau_1-\tau_0}^\lambda, 0), \vartheta_{\tau_1,t}^\lambda(i) = 1\}.$$

Then for all $\delta > 0$,

$$\lim_{\lambda \rightarrow 0} \mathbb{P} [|\Xi_{\tau_0,\tau_1}^\lambda - (\tau_1 - \tau_0)| \geq \delta] = 0.$$

This has been checked in Step 1 of the proof of Lemma 7.2 when $\tau_0 = 0$. This of course extends without any difficulty, using time stationarity.

Step 2. Assume $t_0 = 0$. In that case, the process not depends on i_0 . Consider the event $\Omega_{\lambda,\pi}^{P,T}(0, t_1)$, recall Definition 4.7. We define

$$\begin{aligned} \tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}} &= \Omega_{\lambda,\pi}^{P,T}(0, t_1) \cap \{\exists i_1 \in \llbracket -\mathbf{m}_\lambda, 0 \rrbracket, N_{\mathbf{a}_\lambda(t_1+\kappa_{\lambda,\pi}^0)}^S(i_1) = 0\} \\ &\quad \cap \{\exists i_2 \in \llbracket 0, \mathbf{m}_\lambda \rrbracket, N_{\mathbf{a}_\lambda(t_1+\kappa_{\lambda,\pi}^0)}^S(i_2) = 0\}. \end{aligned}$$

Lemma 4.2 together with Lemma 8.1-1 show that $\mathbb{P} [\tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}}]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$ (because $t_1 + \kappa_{\lambda,\pi}^0 < (t_1 + 1)/2 < 1$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$).

Next, on $\tilde{\Omega}_{\lambda,\pi}^{P,T}(0, t_1)$, there holds that

$$C(\vartheta_{0,t_1+\kappa_{\lambda,\pi}^0}^\lambda, 0) := \llbracket C^-, C^+ \rrbracket \subset \llbracket i_1, i_2 \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket.$$

Since, by definition, no seed falls on C^+ and on C^- until $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)$ and since we start from a vacant initial situation, we deduce that

$$\zeta_t^{\lambda,\pi,\mathcal{M}}(C^-) = \zeta_t^{\lambda,\pi,\mathcal{M}}(C^+) = 0$$

for all $t \in [0, \mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)] \supset [\mathbf{a}_\lambda t_1, \mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)]$. As seen in **Micro**(p) in Subsection 4.4, the fire destroys exactly the zone $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ and

$$C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) \subset \llbracket C^-, C^+ \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$$

with $\zeta_{\mathbf{a}_\lambda(t_1+\kappa_{\lambda,\pi}^0)}^{\lambda,\pi,\mathcal{M}}(i) \leq 1$ for all $i \in \mathbb{Z}$ (the fire is extinguished at time $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)$).

Since $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ clearly contains $C(\vartheta_{0,t_1}^\lambda, 0)$, we deduce that, on $\tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}}$,

$$t_1 + \Xi_{0,t_1}^\lambda \leq t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi} \leq t_1 + \kappa_{\lambda,\pi}^0 + \Xi_{0,t_1+\kappa_{\lambda,\pi}^0}^\lambda.$$

Remark now that the function $t \mapsto t + \Xi_{0,t}^\lambda$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi} \xrightarrow[\lambda,\pi]{\mathbb{P}} 2t_1$$

in probability, whence for all $\delta > 0$ and all $\varepsilon > 0$, there holds that $\mathbb{P} [|\Theta_{\mathcal{M}}^{\lambda,\pi} - t_1| \geq \delta] < \varepsilon$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

Step 3. Assume now $t_0 > 1$. We may and will assume $i_0 \in \llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\mathbf{m}_\lambda \rrbracket$, by symetry.

Consider the events $\Omega_{\lambda,\pi}^{P,T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$ and $\Omega_{\lambda,\pi}^{P,T}(0, t_1)$, recall Definition 4.7. We define

$$\begin{aligned} \tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}} &:= \Omega_{\lambda,\pi}^{P,T}(0, t_1) \cap \Omega_{\lambda,\pi}^{P,T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi}) \\ &\quad \cap \{\exists i_1 \in \llbracket -\mathbf{m}_\lambda, 0 \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)}^S(i_1) - N_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})}^S(i_1) = 0\} \\ &\quad \cap \{\exists i_2 \in \llbracket 0, \mathbf{m}_\lambda \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)}^S(i_2) - N_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})}^S(i_2) = 0\}. \end{aligned}$$

Lemma 4.2 together with Lemma 8.1-1 directly imply that $\mathbb{P}[\tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}}]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$ (because $t_1 + \kappa_{\lambda,\pi}^0 - (t_0 - \mathbf{v}_{\lambda,\pi}) = t_1 - t_0 + \kappa_{\lambda,\pi}^0 + \mathbf{v}_{\lambda,\pi} < (t_1 - t_0 + 1)/2 < 1$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$).

Recall Lemma 4.2. Since all the sites are occupied at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$ and since

$$i_0 + \lfloor \mathbf{a}_\lambda \pi (3\mathbf{v}_{\lambda,\pi} - \varepsilon_\lambda) \rfloor \geq \mathbf{m}_\lambda,$$

on $\Omega_{\lambda,\pi}^{P,T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$, there is no more burning tree in $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})$ nor during the time interval $[\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda t_1]$. Thus, the match falling in 0 at time $\mathbf{a}_\lambda t_1$ destroys at least the zone $C(\vartheta_{t_0+2\mathbf{v}_{\lambda,\pi}, t_1}^\lambda, 0)$.

Next, on $\tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}}$, we have

$$C(\vartheta_{t_0 - \mathbf{v}_{\lambda,\pi}, t_1 + \kappa_{\lambda,\pi}^0}^\lambda, 0) := \llbracket C^-, C^+ \rrbracket \subset \llbracket i_1, i_2 \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket.$$

Since no seed falls on C^- and on C^+ during $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)]$ and since C^- and C^+ are made vacant during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})]$, thanks to $\Omega_{\lambda,\pi}^{P,T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$, we deduce that there is no burning tree in $\llbracket C^-, C^+ \rrbracket$ at time $\mathbf{a}_\lambda t_1$ – and

$$\zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi,\mathcal{M}}(C^-) = \zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi,\mathcal{M}}(C^+) = 0 \text{ for all } t \in [t_1, t_1 + \kappa_{\lambda,\pi}^0].$$

Hence, as seen in **Micro**(p) in Subsection 4.4, the match falling on 0 at time $\mathbf{a}_\lambda t_1$ destroys at most the zone $\llbracket C^-, C^+ \rrbracket \subset \llbracket i_1, i_2 \rrbracket$ and there is no more burning tree in $\llbracket C^-, C^+ \rrbracket$ at time $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)$.

To summarize, on $\tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}}$, see Figure 6, we have

$$C(\vartheta_{t_0+2\mathbf{v}_{\lambda,\pi}, t_1}^\lambda, 0) \subset C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) \subset C(\vartheta_{t_0 - \mathbf{v}_{\lambda,\pi}, t_1 + \kappa_{\lambda,\pi}^0}^\lambda, 0) \subset \llbracket i_1, i_2 \rrbracket$$

with additionally $\zeta_{\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)}^{\lambda,\pi,\mathcal{M}}(i) \leq 1$ for all $i \in \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$.

Since no fire affect the zone $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ during $[\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0), \mathbf{a}_\lambda T]$, thanks to $\Omega_{\lambda,\pi}^{P,T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$, we deduce that, on $\tilde{\Omega}_{\lambda,\pi}^{P,T,\mathcal{M}}$ and for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$,

$$t_1 + \Xi_{t_0+2\mathbf{v}_{\lambda,\pi}, t_1}^\lambda \leq t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi} \leq t_1 + \kappa_{\lambda,\pi}^0 + \Xi_{t_0 - \mathbf{v}_{\lambda,\pi}, t_1 + \kappa_{\lambda,\pi}^0}^\lambda.$$

Then, one easily concludes. The function $s \mapsto t_1 + \Xi_{t_0+s, t_1}^\lambda$ is a.s. non increasing and right-continuous while the function $s \mapsto t_1 + s + \Xi_{t_0-s, t_1+s}^\lambda$ is a.s. non decreasing and right-continuous. Since $\kappa_{\lambda,\pi}^0 \rightarrow 0$, we thus deduce from Step 1 that

$$t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi} \xrightarrow[\lambda,\pi]{\mathbb{P}} 2t_1 - t_0,$$

as desired. \square

8.3 Persistent effect of microscopic fires

Here we study the effect of microscopic fires. First, they produce a barrier, and then, if there are alternatively macroscopic fires on the left and right, they still have an effect. This phenomenon is illustrated on Figure 7 in the case of the limit process.

We say that $\mathcal{P} = (t_0, t_1, \dots, t_K)$ satisfies (PP1) (like ping-pong) if

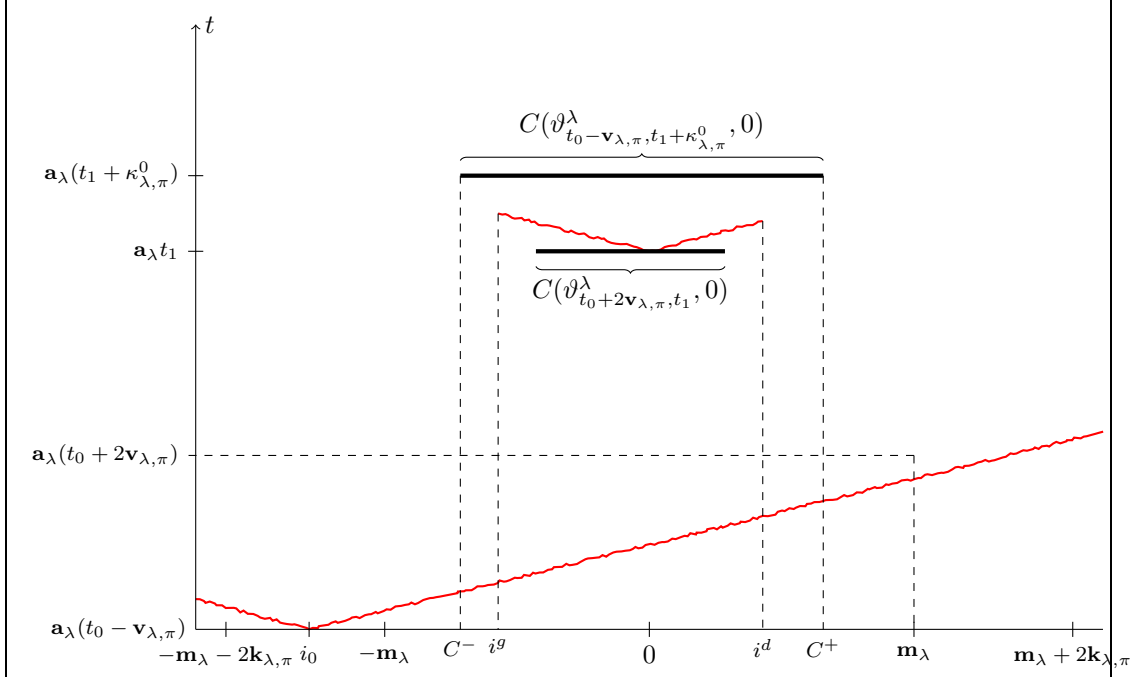


Figure 6: Height of a barrier in the regime $\mathcal{R}(p)$, for $p > 0$.

At time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})-$, all the sites are occupied. A match falls on i_0 at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$. Two fires start: one goes to the left and one goes to the right. Thus, on $\Omega_{\lambda,\pi}^{P,T}(i_0/\mathbf{n}_\lambda, t_0 - \mathbf{v}_{\lambda,\pi})$, each site of $\llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$ burns and extinguishes before $\mathbf{a}_\lambda(t_0 + 2\mathbf{v}_{\lambda,\pi})$ (because $i_0 + \lfloor \mathbf{a}_\lambda \pi(3\mathbf{v}_{\lambda,\pi} - \varepsilon_\lambda) \rfloor \geq \mathbf{m}_\lambda$). Next, a match falls on 0 at time $\mathbf{a}_\lambda t_1$. Since no seed fall on $C^- \in \llbracket -\mathbf{m}_\lambda, 0 \rrbracket$ and $C^+ \in \llbracket 0, \mathbf{m}_\lambda \rrbracket$ during $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)]$, they remain vacant after burning. Thus, the true killed cluster $\llbracket i^g, i^d \rrbracket$ contains $C(\vartheta_{t_0 + 2\mathbf{v}_{\lambda,\pi}, t_1}^\lambda, 0)$ but is included in $\llbracket C^-, C^+ \rrbracket = C^P((\zeta_t^{\lambda,\pi,\mathcal{M}})_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$.

1. $K \geq 2$;
2. $t_0 \in \{0\} \cup (1, \infty)$ and $t_0 < t_1 < t_2 < \dots < t_K$;
3. for all $k = 0, \dots, K-1$, $t_{k+1} - t_k < 1$;
4. $t_2 - t_0 > 1$ and for all $k = 2, \dots, K-2$, $t_{k+2} - t_k > 1$.

We say that $\mathcal{I} = (\varepsilon; i_0, i_2, \dots, i_K)$ satisfies (PP2) if

1. $\varepsilon \in \{-1, 1\}$;
2. $|i_0| \in \llbracket \mathbf{m}_\lambda, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$;
3. for all $k = 2, \dots, K$, $\varepsilon_k i_k \in \llbracket \mathbf{m}_\lambda, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$, where we set $\varepsilon_k = (-1)^k \varepsilon$.

Finally, we say that $\mathfrak{P} = (\mathcal{P}, \mathcal{I})$ satisfies (PP) if \mathcal{P} satisfies (PP1) and \mathcal{I} satisfies (PP2).

Let \mathfrak{P} satisfy (PP). Consider two Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all this processes being independent. We define the process $(\zeta_t^{\lambda,\pi,\mathfrak{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$

as follows

$$\begin{aligned}
\zeta_t^{\lambda, \pi, \mathfrak{P}}(i) = & (1 + \mathbf{1}_{\{i=i_0, t \geq \mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda, \pi})\}}) \mathbf{1}_{\{t_0 \geq 1\}} + \mathbf{1}_{\{i=0, t \geq \mathbf{a}_\lambda t_1, \zeta_{\mathbf{a}_\lambda t_1 -}^{\lambda, \pi, \mathfrak{P}}(0)=1\}} \\
& + \sum_{k=2}^K \mathbf{1}_{\{i=i_k, t \geq \mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda, \pi}), \zeta_{\mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda, \pi}) -}^{\lambda, \pi, \mathfrak{P}}(i_k)=1\}} \\
& + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathfrak{P}}(i)=0\}} dN_s^S(i) \\
& + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathfrak{P}}(i-1)=2, \zeta_{s-}^{\lambda, \pi, \mathfrak{P}}(i)=1\}} dN_s^P(i-1) + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathfrak{P}}(i+1)=2, \zeta_{s-}^{\lambda, \pi, \mathfrak{P}}(i)=1\}} dN_s^P(i+1) \\
& - 2 \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathfrak{P}}(i)=2\}} dN_s^P(i).
\end{aligned}$$

We now explain the behaviour of the process $(\zeta_t^{\lambda, \pi, \mathfrak{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$.

- If $t_0 = 0$, then the process starts from a vacant initial configuration. The match falling on 0 at time $\mathbf{a}_\lambda t_1 \in (0, \mathbf{a}_\lambda)$ creates a barrier, see Lemma 8.2, because $t_1 \in (0, 1)$. Then, fires start in i_k alternately on the right and on the left of 0 at times $\mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda, \pi})$ for all $k = 2, \dots, K$ and fires spread accross \mathbb{Z} according to the same rules as the (λ, π) -FFP.
- If $t_0 > 1$, the process starts from an occupied initial situation. Nothing happens until a match falls on i_0 at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda, \pi})$ and spreads across \mathbb{Z} . Next, a match falls on 0 at time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + 1))$. It then creates a barrier, see Lemma 8.2. Afterwards, matches fall successively in i_k at time $\mathbf{a}_\lambda(t_k - \mathbf{v}_{\lambda, \pi})$ for each $k = 2, \dots, K$ and fires spread accross \mathbb{Z} according to the same rules as the (λ, π) -FFP.

Consider the event

$$\Omega_{\mathfrak{P}}^{S, P}(\lambda, \pi) = \{\forall k \in \{2, \dots, K\}, \exists j \in \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket, \forall t \in [t_k + 2\mathbf{v}_{\lambda, \pi}, t_k + 1 - \mathbf{v}_{\lambda, \pi}), \zeta_{\mathbf{a}_\lambda t}^{\lambda, \pi, \mathfrak{P}}(j) = 0\}.$$

Lemma 8.3. *Let $\mathcal{P} = (t_0, \dots, t_K)$ satisfy (PP1) and $\mathcal{I} = (\varepsilon; i_0, i_2, \dots, i_K)$ satisfy (PP2). For each $\lambda \in (0, 1)$ and each $\pi \geq 1$, consider the process $(\zeta_t^{\lambda, \pi, \mathfrak{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$ defined above.*

If $t_2 - t_1 < t_1 - t_0$, when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, there holds

$$\lim_{\lambda, \pi} \mathbb{P} \left[\Omega_{\mathfrak{P}}^{S, P}(\lambda, \pi) \right] = 1.$$

Proof. We define, recall Definition 4.7,

$$\Omega_{\lambda, \pi}^{P, T, \mathfrak{P}} = \Omega_{\lambda, \pi}^{P, T}(0, t_1) \cap \bigcap_{k=0, 2, \dots, K} \Omega_{\lambda, \pi}^{P, T} \left(\frac{i_k}{\mathbf{n}_\lambda}, t_k - \mathbf{v}_{\lambda, \pi} \right).$$

There holds that $\mathbb{P} \left[\Omega_{\lambda, \pi}^{P, T, \mathfrak{P}} \right]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, by Lemma 4.2.

In the whole proof, we work on $\Omega_{\lambda, \pi}^{P, T, \mathfrak{P}}$ and assume that (λ, π) is sufficiently close to the regime $\mathcal{R}(p)$ in such a way that $3\mathbf{v}_{\lambda, \pi} < \min_{i=1, \dots, K} (t_{i+1} - t_i) < 1 - 3\mathbf{v}_{\lambda, \pi}$.

For simplicity, we assume that $\varepsilon = -1$, $t_0 = 0$ and that K is even (see for example Step 3 in Lemma 8.2. The other cases are treated similarly. Fix $\alpha = 1/K$. We define $\mathcal{M} = (0; 0, t_1)$, recall Lemma 8.2.

Observe that on $\Omega_{\lambda, \pi}^{P, T, \mathfrak{P}}$, a burning tree at time $\mathbf{a}_\lambda t$ necessarily belongs to $\llbracket i_k + \lfloor \mathbf{a}_\lambda \pi(t - t_k - \varepsilon_\lambda) \rfloor, i_k + \lfloor \mathbf{a}_\lambda \pi(t - t_k + \varepsilon_\lambda) \rfloor \rrbracket$ or to $\llbracket i_k - \lfloor \mathbf{a}_\lambda \pi(t - t_k + \varepsilon_\lambda) \rfloor, i_k - \lfloor \mathbf{a}_\lambda \pi(t - t_k - \varepsilon_\lambda) \rfloor \rrbracket$, for some $k \in \{0, \dots, K\}$ and is either a front of a fire or has vacant neighbors.

Observe that for all $i \in \llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, -\mathbf{m}_\lambda \rrbracket$, we have, recall (8.1) and (8.2),

$$i + \lfloor \mathbf{a}_\lambda \pi(3\mathbf{v}_{\lambda, \pi} - \varepsilon_\lambda) \rfloor \geq \mathbf{m}_\lambda \tag{8.7}$$

whence for all $i \in \llbracket \mathbf{m}_\lambda, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi} \rrbracket$, we have

$$i - \lfloor \mathbf{a}_\lambda \pi(3\mathbf{v}_{\lambda, \pi} - \varepsilon_\lambda) \rfloor \leq -\mathbf{m}_\lambda. \tag{8.8}$$

First fire. We put $C^P = C^P((\zeta_t^{\lambda,\pi,\mathfrak{P}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$, the destroyed cluster due to the match falling on 0 at time $\mathbf{a}_\lambda t_1$, recall (4.12). Since $0 < t_1 < 1$, there holds $C^P \subset \llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ with probability tending to 1 (use Lemma 8.1-1, space/time stationarity and **Micro**(p) in Subsection 4.4). Thus the match falling at time $\mathbf{a}_\lambda t_1$ destroys nothing outside $\llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ and there is no more burning tree in \mathbb{Z} at time $\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0)$.

Second fire. Since $t_2 - \mathbf{v}_{\lambda,\pi} > 1$, at least one seed has fallen, during $[0, \mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}))$, on each site of $\llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ with probability tending to 1 (use Lemma 8.1-2 and space/time stationarity). Since this zone has not been affected by a fire during the time interval $[0, \mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}))$, this zone is completely occupied at time $\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi})$.

Besides, with probability tending to 1, there is (at least) an empty site in $C^P \subset \llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during the time interval $(\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0), \mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi}))$ because $t_2 + 2\mathbf{v}_{\lambda,\pi} < t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi}$ with probability tending to 1 (by Lemma 8.2, $\Theta_{\mathcal{M}}^{\lambda,\pi} \simeq t_1 - t_0 = t_1$ and $t_2 - t_1 < t_1 - t_0 = t_1$ by assumption) and because by definition of $\Theta_{\mathcal{M}}^{\lambda,\pi}$, there is an empty site in $C^P \subset \llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during $[\mathbf{a}_\lambda(t_1 + \kappa_{\lambda,\pi}^0), \mathbf{a}_\lambda(t_1 + \Theta_{\mathcal{M}}^{\lambda,\pi})]$.

Thus, the fire ignited on $i_2 \in \llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi})$ burns each site of $\llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ before $\mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$, thanks to (8.7) and $\Omega_{\lambda,\pi}^{P,T}(i_2/\mathbf{n}_\lambda, t_2 - \mathbf{v}_{\lambda,\pi})$ (because the right front of the fire 2 reach a vacant site and thus extinguish).

Third fire. All the sites of $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$ are occupied at time $\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi})$ with probability tending to 1 (because on $\Omega_{\lambda,\pi}^{P,T}(0, t_1) \cap \Omega_{\lambda,\pi}^{P,T}(i_2/\mathbf{n}_\lambda, t_2 - \mathbf{v}_{\lambda,\pi})$, they have not been affected by a fire during $[0, \mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi}))$ and because $t_3 - \mathbf{v}_{\lambda,\pi} > t_2 - \mathbf{v}_{\lambda,\pi} > 1$, see Lemma 8.1-2.).

Next, the probability that there is a site in $\llbracket -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ where no seed falls during $[\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi} + 1)]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$ (use Lemma 8.1-1 and space/time stationarity). Thus, since $t_3 - t_2 < 1$, with probability tending to 1, there exists a vacant site in $\llbracket -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during

$$[\mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi} + 1)] \supset [\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_3 + 2\mathbf{v}_{\lambda,\pi})]$$

(because each site of $\llbracket -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ has been made vacant by the second fire during $[\mathbf{a}_\lambda(t_2 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi})]$).

Thus, the fire ignited on $i_3 \in \llbracket \mathbf{m}_\lambda, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$ at time $\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi})$ burns each site of $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$ before $\mathbf{a}_\lambda(t_3 + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $\llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ with probability tending to 1, thanks to (8.8) and $\Omega_{\lambda,\pi}^{P,T}(i_3/\mathbf{n}_\lambda, t_3 - \mathbf{v}_{\lambda,\pi})$ (because the left front of the fire 3 reach a vacant site and thus extinguish).

Fourth fire. All the sites of $\llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ are occupied at time $\mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi})$ with probability tending to 1 (because on $\Omega_{\lambda,\pi}^{P,T}(0, t_1) \cap \Omega_{\lambda,\pi}^{P,T}(i_2/\mathbf{n}_\lambda, t_2 - \mathbf{v}_{\lambda,\pi}) \cap \Omega_{\lambda,\pi}^{P,T}(i_3/\mathbf{n}_\lambda, t_3 - \mathbf{v}_{\lambda,\pi})$, they have not been affected by a fire during $(\mathbf{a}_\lambda(t_2 + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi}))$ and because $t_4 - 3\mathbf{v}_{\lambda,\pi} - t_2 > 1$, see Lemma 8.1-2 and spae/time stationarity).

The probability that there is a site in $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor 2\alpha \mathbf{m}_\lambda \rfloor \rrbracket$ where no seed falls during $[\mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi} + 1)]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$ (use Lemma 8.1-1 and space/time stationarity). Hence, since $t_4 - t_3 < 1$, there is at least one vacant site in $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor 2\alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during

$$[\mathbf{a}_\lambda(t_3 + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_3 - \mathbf{v}_{\lambda,\pi} + 1)] \supset [\mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_4 + 2\mathbf{v}_{\lambda,\pi})],$$

with probability tending to 1.

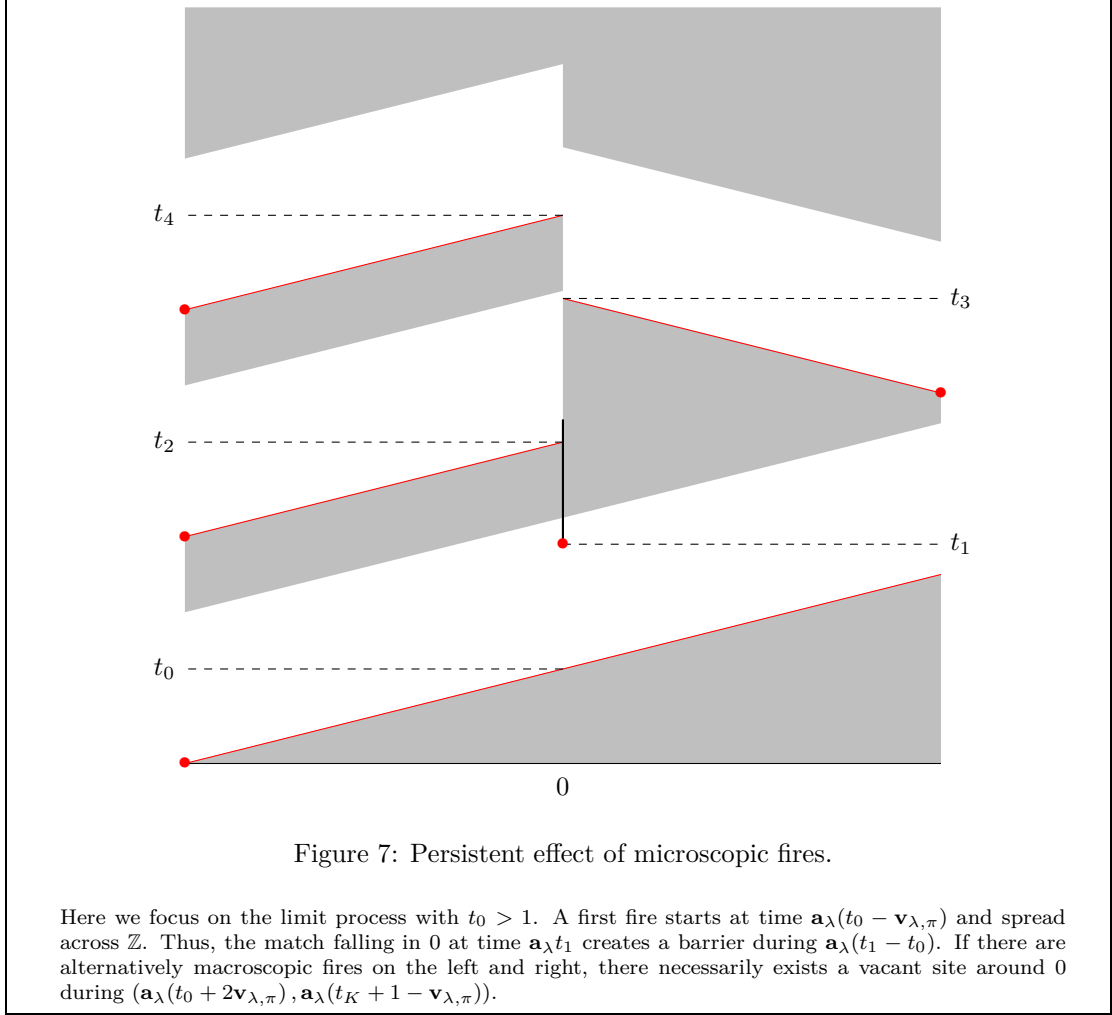
Thus, the fire ignited on $i_4 \in \llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda(t_4 - \mathbf{v}_{\lambda,\pi})$ burns each site of $\llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ before $\mathbf{a}_\lambda(t_4 + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$ with probability tending to 1, thanks to (8.7) and $\Omega_{\lambda,\pi}^{P,T}(i_4/\mathbf{n}_\lambda, t_4 - \mathbf{v}_{\lambda,\pi})$.

Last fire and conclusion. Iterating the procedure, we see that with a probability tending to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, the zone $\llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor (K\alpha/2)\mathbf{m}_\lambda \rfloor - 1 \rrbracket = \llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \mathbf{m}_\lambda/2 \rfloor - 1 \rrbracket$ is completely occupied at time $\mathbf{a}_\lambda(t_K - \mathbf{v}_{\lambda,\pi})$ and there is at least one vacant site in $\llbracket \lfloor (K-1)\alpha/2\mathbf{m}_\lambda \rfloor, \lfloor (K\alpha/2)\mathbf{m}_\lambda \rfloor \rrbracket$ during the time interval $(\mathbf{a}_\lambda(t_{K-1} + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_{K-1} -$

$\mathbf{v}_{\lambda,\pi}+1)) \supset (\mathbf{a}_{\lambda}(t_K - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_{\lambda}(t_K + 2\mathbf{v}_{\lambda,\pi}))$. Thus, the fire ignited on $i_K \in \llbracket -\mathbf{m}_{\lambda} - 2\mathbf{k}_{\lambda,\pi}, -\mathbf{m}_{\lambda} \rrbracket$ at time $\mathbf{a}_{\lambda}(t_K - \mathbf{v}_{\lambda,\pi})$ destroys each site of the zone $\llbracket -\mathbf{m}_{\lambda} - 2\mathbf{k}_{\lambda,\pi}, -\lfloor \mathbf{m}_{\lambda}/2 \rfloor - 1 \rrbracket$ before $\mathbf{a}_{\lambda}(t_K + 2\mathbf{v}_{\lambda,\pi})$ and does not affect the zone $\llbracket \mathbf{m}_{\lambda}/2, \mathbf{m}_{\lambda} \rrbracket$, thanks to (8.7) and $\Omega_{\mathbf{a}_{\lambda},\pi}^{P,T}(i_K/\mathbf{n}_{\lambda}, t_K - \mathbf{v}_{\lambda,\pi})$.

Finally, the probability that there is at least one site in $\llbracket -\mathbf{m}_{\lambda}, -\mathbf{m}_{\lambda}/2 \rrbracket$ with no seed falling during $[\mathbf{a}_{\lambda}(t_K - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_{\lambda}(t_K - \mathbf{v}_{\lambda,\pi} + 1)]$ tends to 1 (by Lemma 8.1-1.). Consequently, the probability that there is a vacant site in $\llbracket -\mathbf{m}_{\lambda}, -\mathbf{m}_{\lambda}/2 \rrbracket$ during $[\mathbf{a}_{\lambda}(t_K + 2\mathbf{v}_{\lambda,\pi}), \mathbf{a}_{\lambda}(t_K - \mathbf{v}_{\lambda,\pi} + 1)]$ tends to 1 (because it has been made vacant by the fire K).

All this implies that for all $k \in \{2, \dots, K\}$, there is $j \in \llbracket -\mathbf{m}_{\lambda}, \mathbf{m}_{\lambda} \rrbracket$ such that for all $t \in [t_k + 2\mathbf{v}_{\lambda,\pi}, t_k + 1 - \mathbf{v}_{\lambda,\pi})$ there holds $\zeta_{\mathbf{a}_{\lambda}t}^{\lambda,\pi,\mathfrak{P}}(j) = 0$, as desired. \square



8.4 Heart of the proof

8.4.1 The coupling

We are going to construct a coupling between the (λ, π, A) -FFP (on the time interval $[0, \mathbf{a}_{\lambda}T]$) and the A -LFFP(p) (on $[0, T]$). Let π_M be a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$.

First, we take for the matches of the discrete process the Poisson processes

$$N_t^M(i) = \pi_M([i/\mathbf{n}_{\lambda}, (i+1)/\mathbf{n}_{\lambda}) \times [0, t/\mathbf{a}_{\lambda}])$$

for all $i \in \mathbb{Z}$ and $t \in [0, T]$.

We call $n := \pi_M([0, T] \times [-A, A])$ and we consider the marks $(T_q, X_q)_{q=1, \dots, n}$ of π_M ordered in such a way that $0 < T_1 < \dots < T_n < T$.

Next, we introduce some i.i.d. families of i.i.d. Poisson processes $(N_t^{S,q}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^{P,q}(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective parameter 1 and π , for $q = 0, 1, \dots$, independent of π_M .

Then we build two families of i.i.d. Poisson processes $(N_t^{S,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^{P,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ as follows.

- For $q \in \{1, \dots, n\}$, for all $i \in [X_q]_{\lambda,\pi}$, set $(N_t^{S,\lambda,\pi}(i))_{t \geq 0} = (N_t^{S,q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0}$ and $(N_t^{P,\lambda,\pi}(i))_{t \geq 0} = (N_t^{P,q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0}$ (if i belongs to $[X_q]_{\lambda,\pi} \cap [X_r]_{\lambda,\pi}$ for some $q < r$, set *e.g.* $(N_t^{S,\lambda,\pi}(i))_{t \geq 0} = (N_t^{S,q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0}$ and $(N_t^{P,\lambda,\pi}(i))_{t \geq 0} = (N_t^{P,q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0}$. This will occur with a very small probability, so that this choice is not important).
- For all other $i \in \mathbb{Z}$ set $(N_t^{S,\lambda,\pi}(i))_{t \geq 0} = (N_t^{S,0}(i))_{t \geq 0}$ and $(N_t^{P,\lambda,\pi}(i))_{t \geq 0} = (N_t^{P,0}(i))_{t \geq 0}$.

The (λ, π, A) -FFP $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in I_A^\lambda}$ is built from the seed processes $(N_t^{S,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, the match processes $(N_t^M(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^{P,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Finally, we build the A -LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t \in [0, T], x \in [-A, A]}$ from π_M and observe that it is independent of $(N_t^{S,q}(i))_{t \in [0, \mathbf{a}_\lambda T], i \in \mathbb{Z}, q \geq 0}$ and $(N_t^{P,q}(i))_{t \in [0, \mathbf{a}_\lambda T], i \in \mathbb{Z}, q \geq 0}$.

Observe that if a match falls at some X_q at time T_q for the LFFP(p), it will fall at $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$ in the discrete process, and thus if the resulting fire is microscopic in the limit process, it will involve with the same seed and propagation processes for all values of λ and π in discrete process.

8.4.2 A favorable event

We set $T_0 = 0$ and introduce

$$\mathcal{T}_M = \{T_0, T_1, \dots, T_n\} \text{ and } \mathcal{B}_M = \{X_1, \dots, X_n\}.$$

For $q \in \{1, \dots, n\}$, $x \in [-A, A]$ and $t \in [0, T]$, we define

$$T_q(x) = T_q + p|x - X_q| \quad (8.9)$$

$$X_q^+(t) = X_q + \frac{t - T_q}{p} \quad (8.10)$$

$$X_q^-(t) = X_q - \frac{t - T_q}{p} \quad (8.11)$$

which are respectively the possible transit time in x of the fire starting in X_q at time T_q and the possible location of the right and the left front at time t of the fire starting in X_q at time T_q . Observe that all $x \in [-A, A]$ either equal to $X_k^+(T_k(x))$ or $X_k^-(T_k(x))$.

We next introduce, for $q \in \{1, \dots, n\}$,

$$\mathcal{S}_{M,q} = \{T_k(X_q) = T_k + p|X_q - X_k| : k \neq q\}$$

the set of all the possible transit times in X_q of the other fire k and

$$\mathcal{S}_M = \cup_{q=1, \dots, n} \mathcal{S}_{M,q}.$$

We also introduce

$$\mathcal{S}_M^1 = \{2T_q - s : q \in \{1, \dots, n\}, s \in \mathcal{S}_{M,q}, s < T_q\}$$

which has to be seen as the set of the possible end of the microscopic fires, recall Lemma 8.2 and, for $q \in \{2, \dots, n\}$,

$$\mathcal{B}_{M,q}^1 = \left\{ X_k^+(T_q) = X_k + \frac{T_q - T_k}{p} : 1 \leq k < q \right\} \cup \left\{ X_k^-(T_q) = X_k + \frac{T_q - T_k}{p} : 1 \leq k < q \right\}$$

which has to be seen as the set of the possible locations of the fire k at time T_q .

We finally introduce

$$\mathcal{B}_M^2 = \left\{ \frac{T_q - T_k}{2p} + \frac{X_q + X_k}{2} : X_k < X_q \right\} \text{ and } \mathcal{S}_M^2 = \left\{ \frac{T_q + T_k}{2} + p \frac{X_q + X_k}{2} : 1 \leq k < q \leq n \right\}$$

which has to be seen as the set of the possible locations and the set of the possible times where two fires may meet as well as the set \mathcal{C}_M of connected component of $[-A, A] \setminus (\mathcal{B}_M \cup \mathcal{B}_M^2)$ (sometimes refers as cells).

For $\alpha > 0$, we consider the event

$$\Omega_M(\alpha) = \left\{ \begin{array}{l} \min_{\substack{s, t \in \mathcal{T}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2, \\ s \neq t}} |t - s| \geq 3\alpha, \quad \min_{s, t \in \mathcal{T}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2} |t - (s + 1)| \geq 3\alpha, \\ \min_{\substack{x, y \in \mathcal{B}_M \cup \mathcal{B}_M^2 \cup \{-A, A\}, \\ x \neq y}} |x - y| \geq \frac{3\alpha}{p} \end{array} \right\}$$

which clearly satisfies $\lim_{\alpha \rightarrow 0} \mathbb{P}[\Omega_M(\alpha)] = 1$. For any given $\alpha > 0$, there exists $\lambda_\alpha \in (0, 1)$ and $\varepsilon_\alpha > 0$ such that for all $\lambda \in (0, \lambda_\alpha)$ and all $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda / (\mathbf{a}_\lambda \pi) - p| < \varepsilon_\alpha$, on $\Omega_M(\alpha)$, there holds that for all $x, y \in \mathcal{B}_M \cup \mathcal{B}_M^2 \cup \{-A, A\}$, with $x \neq y$, $[x]_{\lambda, \pi} \cap [y]_{\lambda, \pi} = \emptyset$.

For $q \in \{1, \dots, n\}$, using the seed processes $(N_t^{S, \lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^{P, \lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, we build, recall Definition 4.6, $(\zeta_t^{\lambda, \pi, q}(i))_{t \geq 0, i \in \mathbb{Z}}$ (the propagation process ignited at (X_q, T_q)), $(i_t^{q, +})_{t \geq 0}$ and $(i_t^{q, -})_{t \geq 0}$ (the corresponding right and left fronts) and $(T_i^q)_{i \in \mathbb{Z}}$ (the associated burning times). We also use $\Omega_{\lambda, \pi}^{P, T}(X_q, T_q)$, recall Definition 4.7. We set

$$\Omega^{P, T}(\lambda, \pi) = \bigcap_{q=1, \dots, n} \Omega_{\lambda, \pi}^{P, T}(X_q, T_q).$$

Since π_M is independent of the processes $(N_t^{S, \lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^{P, \lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$, Lemma 4.2 implies that $\mathbb{P}[\Omega^{P, T}(\lambda, \pi)]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

Let $q \in \{1, \dots, n\}$. We define

$$\mathcal{I}^{q, +} := \left\{ \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_k(X_q) - \mathbf{v}_{\lambda, \pi} - T_k)}^{k, +} - \lfloor \mathbf{n}_\lambda X_k^+(T_k(X_q)) \rfloor : k \neq q \right\} \quad (8.12)$$

$$\mathcal{I}^{q, -} := \left\{ \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_k(X_q) - \mathbf{v}_{\lambda, \pi} - T_k)}^{k, -} - \lfloor \mathbf{n}_\lambda X_k^-(T_k(X_q)) \rfloor : k \neq q \right\}. \quad (8.13)$$

Observe that, on $\Omega^{P, T}(\lambda, \pi)$, $\mathcal{I}^{q, -} \subset \llbracket \mathbf{m}_\lambda, \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi} \rrbracket$ whence $\mathcal{I}^{q, +} \subset \llbracket -\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, -\mathbf{m}_\lambda \rrbracket$. We then call \mathcal{U}_q the set of all possible $\mathfrak{P} = (\mathcal{P}, \mathcal{I})$ satisfying (PP) where

- $\mathcal{P} = (t_0, T_q, t_2, \dots, t_K)$ satisfies $(PP1)$ with $\{t_0, t_2, \dots, t_K\} \subset \mathcal{S}_{M, q} \cup \{0\}$ and with $T_q - t_0 > t_2 - T_q$;
- $\mathcal{I} = (\varepsilon; i_0, i_2, \dots, i_K)$ satisfies $(PP2)$ with $\varepsilon \in \{-1, 1\}$ and $\{i_0, i_2, \dots, i_K\} \subset \mathcal{I}^{q, +} \cup \mathcal{I}^{q, -}$.

For $\mathfrak{P} \in \mathcal{U}_q$, we introduce the event $\Omega_{\mathfrak{P}}^{S, P, q}(\lambda, \pi)$, defined as in Subsection 8.3, with the Poisson processes $(N_t^{S, q}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^{P, q}(i))_{t \geq 0, i \in \mathbb{Z}}$. Then we put

$$\Omega_1^{S, P}(\lambda, \pi) = \bigcap_{q=1}^n \left\{ \text{for all } \mathfrak{P} \in \mathcal{U}_q, \Omega_{\mathfrak{P}}^{S, P, q}(\lambda, \pi) \text{ holds} \right\},$$

which satisfies $\lim_{\lambda, \pi} \mathbb{P}[\Omega_1^{S, P}(\lambda, \pi)] = 1$ when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$. Indeed, by construction, π_M is independent of $(N_t^{S, q}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^{P, q}(i))_{t \geq 0, i \in \mathbb{Z}}$. Observe that for $l \in \{1, \dots, n\}$, the location $i_{\mathbf{a}_\lambda(T_l(X_q) - \mathbf{v}_{\lambda, \pi} - T_l)}^{l, +}$ depends only on the propagation process $N^{P, \lambda, \pi}$ restricted to $[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_l(X_q) - \mathbf{v}_{\lambda, \pi})] \times \mathbb{Z}$ whereas the event $\Omega_{\mathfrak{P}}^{S, P, q}(\lambda, \pi)$ depends on the location only after $\mathbf{a}_\lambda(T_l(X_q) - \mathbf{v}_{\lambda, \pi})$. Thus, it suffices to work with some fixed $\{t_0, t_2, \dots, t_K\} \subset \mathcal{S}_{M, q}$ and some fixed $(i_k)_{k=0, 2, \dots, K} \subset \mathcal{I}^{q, +} \cup \mathcal{I}^{q, -}$. The result then follows from Lemma 8.3.

We also consider the event $\Omega_2^S(\lambda, \pi)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1$ with $0 < t_2 - t_1 < 1$, for all $q = 1, \dots, n$, there are

$$-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} < i_1 < -\mathbf{m}_\lambda < i_2 < 0 < i_3 < \mathbf{m}_\lambda < i_4 < \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}$$

such that $N_{\mathbf{a}_\lambda(t_2+4\mathbf{v}_{\lambda, \pi})}^{S, q}(i_j) - N_{\mathbf{a}_\lambda(t_1-4\mathbf{v}_{\lambda, \pi})}^{S, q}(i_j) = 0$ for $j = 1, \dots, 4$. There holds that $\mathbb{P}[\Omega_2^S(\lambda, \pi)]$ tends to 1 as $\lambda \rightarrow \infty$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$. Indeed, it suffices to prove that almost surely, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\Omega_2^S(\lambda, \pi) \mid \pi_M] = 1$. Since there are a.s. finitely many possibilities for q, t_1, t_2 and since π_M is independent of $(N_t^{S, q}(i))_{t \geq 0, i \in \mathbb{Z}}$, it suffices to work with a fixed $q \in \{1, \dots, n\}$ and some fixed $0 < t_2 - t_1 < 1$. The result then follows from Lemma 8.1-1,8 together with space/time stationarity and the fact that $\mathbf{v}_{\lambda, \pi} \rightarrow 0$.

Next we introduce the event $\Omega_3^S(\lambda, \pi)$ on which the following conditions hold: for all $q \in \{1, \dots, n\}$ and all $i \in I_A^\lambda$

$$N_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)+1+\mathbf{e}_{\lambda, \pi})}^{S, \lambda, \pi}(i) - N_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)+\mathbf{e}_{\lambda, \pi})}^{S, \lambda, \pi}(i) > 0$$

and if $T_q(i/\mathbf{n}_\lambda) \geq 1$,

$$N_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)-4\mathbf{v}_{\lambda, \pi})}^{S, \lambda, \pi}(i) - N_{\mathbf{a}_\lambda(T_q(i/\mathbf{n}_\lambda)-1-4\mathbf{v}_{\lambda, \pi})}^{S, \lambda, \pi}(i) > 0.$$

There holds that $\mathbb{P}[\Omega_3^S(\lambda, \pi)]$ tends to 1 as $\lambda \rightarrow \infty$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$. Observing that $|I_A^\lambda| \simeq 2A\mathbf{n}_\lambda$, Lemma 8.1 and space/time stationarity shows the result.

We also need $\Omega_4^{S, P}(\gamma, \lambda, \pi)$, defined for $\gamma > 0$ as follows: for all $q = 1, \dots, n$, for all $\mathcal{M} = (i_0; t_0, T_q)$ such that $t_0 \in \mathcal{S}_{M, q} \cup \{0\}$ with $t_0 < T_q < t_0 + 1$ and $i_0 \in \mathcal{I}^{q, +} \cup \mathcal{I}^{q, -}$, there holds that $|\Theta_{\mathcal{M}}^{\lambda, \pi, q} - (T_q - t_0)| < \gamma$. Here, $\Theta_{\mathcal{M}}^{\lambda, \pi, q}$ is defined as in Lemma 8.2 with the seed processes family $(N_t^{S, q}(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^{P, q}(i))_{t \geq 0, i \in \mathbb{Z}}$. Lemma 8.2 directly implies that for any $\gamma > 0$, $\mathbb{P}[\Omega_4^{S, P}(\gamma, \lambda, \pi)]$ tends to 1 as $\lambda \rightarrow \infty$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$.

We finally introduce the event

$$\Omega(\alpha, \gamma, \lambda, \pi) = \Omega_M(\alpha) \cap \Omega^{P, T}(\lambda, \pi) \cap \Omega_1^{S, P}(\lambda, \pi) \cap \Omega_2^S(\lambda, \pi) \cap \Omega_3^S(\lambda, \pi) \cap \Omega_4^{S, P}(\gamma, \lambda, \pi).$$

We have shown that for any $\delta > 0$, there exists $\alpha \in (0, 1)$ such that for any $\gamma > 0$, there holds that $\mathbb{P}[\Omega(\alpha, \gamma, \lambda, \pi)] > 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

8.4.3 Heart of the proof

Consider the A -LFFP(p) $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in [-A, A]}$.

For $x \in (-A, A)$, we put

$$\begin{aligned} Z_{t-}(x) &= \lim_{s \nearrow t} Z_s(x), \\ Z_t(x+) &= \lim_{y \searrow x} Z_t(y) \text{ and } Z_t(x-) = \lim_{y \nearrow x} Z_t(y), \\ Z_{t-}(x+) &= \lim_{y \searrow x} Z_{t-p(y-x)-}(y) \text{ and } Z_{t-}(x-) = \lim_{y \nearrow x} Z_{t+p(y-x)-}(y). \end{aligned}$$

For $t \in [0, T]$, we set

$$\begin{aligned} \chi_t^+ &= \{x \in [-A, A] : F_t(x) > 0 \text{ and } Z_t(x+) = 1\}, \\ \chi_t^- &= \{x \in [-A, A] : F_t(x) > 0 \text{ and } Z_t(x-) = 1\}, \\ \chi_t^0 &= \{x \in [-A, A] : H_t(x) > 0 \text{ or } (F_t(x) = 0 \text{ and } Z_t(x+) \neq Z_t(x-))\} \cup \{-A, A\}, \\ \chi_t &= \chi_t^+ \cup \chi_t^- \cup \chi_t^0. \end{aligned}$$

For $x \in \mathcal{B}_M$ and $t \geq 0$ we set

$$\tilde{H}_t(x) = \max(H_t(x), 1 - Z_t(x), 1 - Z_t(x+), 1 - Z_t(x-)). \quad (8.14)$$

Actually, $Z_{t-}(x)$ always equals either $Z_{t-}(x-)$ or $Z_{t-}(x+)$ and these can be distinct only at a point where has occurred a microscopic fire (that is if $x = X_q$ for some $q \in \{1, \dots, n\}$ with $T_q < t$ and $Z_{T_q-}(X_q) < 1$).

For all $x \in (-A, A)$ we define for all $t \in [0, T]$

$$\tau_t(x) = \sup \{s \leq t : F_s(x) > 0 \text{ and } \tilde{H}_{s-}(x) = 0\} \vee 0, \quad (8.15)$$

which represents the last time before t that a macroscopic fire has crossed x . Observe that

$$\text{for } x \notin \mathcal{B}_M, Z_t(x) = \min(t - \tau_t(x), 1) \quad \text{for all } t \in [0, T], \quad (8.16)$$

$$\text{for } q = 1, \dots, n, Z_t(X_q) = \min(t - \tau_t(X_q), 1) \quad \text{for all } t \in [0, T_q]. \quad (8.17)$$

We also define for all $i \in I_A^\lambda$ and all $t \in [0, T]$

$$\rho_t^{\lambda, \pi}(i) = \sup \left\{ s \leq t : \eta_{\mathbf{a}_\lambda s-}^{\lambda, \pi}(i) = 2 \right\} \quad (8.18)$$

where $\mathbf{a}_\lambda \rho_t^{\lambda, \pi}(i)$ represents the last time before $\mathbf{a}_\lambda t$ that the site i has been burnt in the discrete process (with the convention $\eta_{0-}^{\lambda, \pi}(i) = 2$ and $\eta_0^{\lambda, \pi}(i) = 0$ for all $i \in I_A^\lambda$).

For $q \in \{1, \dots, n\}$, we define *the death time of the right front of the q 's fire* as the time where the fire q is stopped in the limit process, that is,

$$T_q^{D,+} = \inf \left\{ t \geq T_q : F_t \left(X_q + \frac{t - T_q}{p} \right) = 0 \right\} \quad (8.19)$$

as well as *the death position of the right front of the q 's fire* as the position where the fire q is stopped in the limit process, that is,

$$X_q^{D,+} = X_q + \frac{T_q^{D,+} - T_q}{p}. \quad (8.20)$$

Similarly, *the death time and position of the left front of the q 's fire* are defined as

$$T_q^{D,-} = \inf \left\{ t \geq T_q : F_t \left(X_q - \frac{t - T_q}{p} \right) = 0 \right\} \quad \text{and} \quad X_q^{D,-} = X_q - \frac{T_q^{D,-} - T_q}{p}.$$

Observe that, if $Z_{T_q-}(X_q) < 1$, then $T_q^{D,-} = T_q = T_q^{D,+}$ and $X_q^{D,+} = X_q = X_q^{D,-}$.

We set

$$\mathcal{B}_M^D := \{X_1^{D,+}, X_1^{D,-}, \dots, X_n^{D,+}, X_n^{D,-}\} \subset \mathcal{B}_M \cup \mathcal{B}_M^2, \quad (8.21)$$

$$\mathcal{T}_M^D := \{T_1^{D,+}, T_1^{D,-}, \dots, T_n^{D,+}, T_n^{D,-}\} \subset \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^2. \quad (8.22)$$

Let $t \in [0, T]$ and $q \in \{1, \dots, n\}$. If $t \in [0, T_q^{D,+} + \mathbf{v}_{\lambda, \pi})$, we set

$$\Omega_{q,t}^{\lambda, \pi, +} = \{ \forall s \in [T_q, (T_q^{D,+} + \mathbf{v}_{\lambda, \pi}) \wedge t], \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,+}) = 2 \}$$

and, if $t \in [T_q^{D,+} + \mathbf{v}_{\lambda, \pi}, T]$, we set

$$\Omega_{q,t}^{\lambda, \pi, +} = \Omega_{q, T_q^{D,+}}^{\lambda, \pi, +} \cap \{ \exists s \in [T_q^{D,+} - \mathbf{v}_{\lambda, \pi}, T_q^{D,+} + \mathbf{v}_{\lambda, \pi}], \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,+}) = 0 \}.$$

Similarly, we set, if $t \in [0, T_q^{D,-} + \mathbf{v}_{\lambda, \pi})$,

$$\Omega_{q,t}^{\lambda, \pi, -} = \{ \forall s \in [T_q, (T_q^{D,-} - \mathbf{v}_{\lambda, \pi}) \wedge t], \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,-}) = 2 \}$$

and, if $t \in [T_q^{D,-} + \mathbf{v}_{\lambda, \pi}, T]$, we set

$$\Omega_{q,t}^{\lambda, \pi, -} = \Omega_{q, T_q^{D,-}}^{\lambda, \pi, -} \cap \{ \exists s \in [T_q^{D,-} - \mathbf{v}_{\lambda, \pi}, T_q^{D,-} + \mathbf{v}_{\lambda, \pi}], \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{\mathbf{a}_\lambda(s-T_q)}^{q,-}) = 0 \}.$$

Finally, we set, for all $t \in [0, T]$,

$$\Omega_t^{\lambda, \pi} = \bigcap_{q=1, \dots, n} \left(\Omega_{q,t}^{\lambda, \pi, +} \cap \Omega_{q,t}^{\lambda, \pi, -} \right).$$

The aim of this section is to prove the following Lemma.

Lemma 8.4. *Let $\alpha > \gamma > 0$. For all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$ in such a way that $4(\mathbf{v}_{\lambda, \pi} + p(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi})/\mathbf{n}_\lambda) \leq \alpha$, $\Omega_T^{\lambda, \pi}$ a.s. holds on $\Omega(\alpha, \gamma, \lambda, \pi)$.*

We work on $\Omega(\alpha, \gamma, \lambda, \pi)$. We fix $\varepsilon_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ such that for all $\lambda \in (0, \lambda_\alpha)$ and all $\pi \geq 1$ in such a way $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) - p| < \varepsilon_\alpha$, we have $\mathbf{v}_{\lambda, \pi} + 3p(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi})/\mathbf{n}_\lambda \leq \alpha$. Observe that for all $x, y \in \mathcal{B}_M \cup \mathcal{B}_M^2 \cup \{-A, A\}$, with $x \neq y$, we then have $[x]_{\lambda, \pi} \cap [y]_{\lambda, \pi} = \emptyset$. Clearly, $\Omega_{T_1}^{\lambda, \pi}$ a.s. holds, because no match falls in I_A^λ before $\mathbf{a}_\lambda T_1$. We will show that for $q = 0, \dots, n-1$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$. This will prove that $\Omega_{T_n}^{\lambda, \pi}$ holds. The extension to $\Omega_T^{\lambda, \pi}$ will be straightforward and will be omitted.

We thus fix $q \in \{0, \dots, n-1\}$ and assume $\Omega_{T_q}^{\lambda, \pi}$. Let \mathcal{A}_q be the set of points where a fire stops during the time interval (T_q, T_{q+1}) that is, $(x, t) \in \mathcal{A}_q$ if $(x, t) = (X_k^{D, +}, T_k^{D, +})$ (or $(X_k^{D, -}, T_k^{D, -})$) for some $k \leq q$ with $T_k^{D, +}$ (or $T_k^{D, -}$) in (T_q, T_{q+1}) . We then put

$$\{(X_q^0, T_q^0), \dots, (X_q^{N_q+1}, T_q^{N_q+1})\} = \mathcal{A}_q \cup \{(X_q, T_q), (X_{q+1}, T_{q+1})\}$$

ordered chronologically (thus $(X_q, T_q) = (X_q^0, T_q^0)$ and $(X_{q+1}, T_{q+1}) = (X_q^{N_q+1}, T_q^{N_q+1})$).

We recall that if $Z_{T_l-}(X_l) = 1$, for some $l \in \{1, \dots, n\}$, on $\Omega_M(\alpha)$, we have by construction,

- $T_l^{D, +} \wedge T_l^{D, -} \geq T_l + 3\alpha$;
- $Z_{T_l-}(y) = 1$ for all $y \in (X_l - 3\alpha/p, X_l + 3\alpha/p)$;
- $F_{T_l(y)}(y) = 1$ and $\tilde{H}_{T_l(y)-}(y) = 0$ for all $y \in (X_l^{D, -}, X_l^{D, +})$;
- for all $t \in [T_l, T_l^{D, +} - 3\alpha]$ and all $y \in (X_l^+(t), X_l^+(t) + 3\alpha/p)$, $\tilde{H}_t(y) = 0$ (similar thing for $X_l^-(t)$);
- for all $t \in [T_l^{D, +} - 3\alpha, T_l^{D, +})$ and all $y \in (X_l^+(t), X_l^+(t) + (T_l^{D, +} - t)/p)$, $Z_t(y) = 1$ (similar thing for $X_l^-(t)$).

Recall that on $\Omega_M(\alpha)$, for all $k \in \llbracket 0, N_q \rrbracket$,

$$T_q^{k+1} - T_q^k > 3\alpha.$$

We decompose the proof in four stages.

- *Stage 0.* We deduce, on $\Omega(\alpha, \gamma, \lambda, \pi)$, the last time that a site has been burned.
- *Stage 1.* We prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_q + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$.
- *Stage 2.* We prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, for $0 \leq k < N_q$, $\Omega_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$ implies $\Omega_{T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$.
- *Stage 3.* We prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$, which is the goal.

In the whole proof, we repeatedly use the following estimates. For $k \in \{1, \dots, n\}$ and $t \geq T_k$, there holds that, recall (8.1), (8.2) and (8.3),

$$\llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - T_k - \varepsilon_\lambda) \rfloor, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - T_k + \varepsilon_\lambda) \rfloor \rrbracket \subset \langle X_k^+(t) \rangle_{\lambda, \pi} \quad (8.23)$$

which is the possible location of the right front of the fire k at time $\mathbf{a}_\lambda t$, recall Lemma 4.2,

$$\begin{aligned} & \llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - \mathbf{v}_{\lambda, \pi} - T_k - \varepsilon_\lambda) \rfloor, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - \mathbf{v}_{\lambda, \pi} - T_k + \varepsilon_\lambda) \rfloor \rrbracket \\ & \subset \llbracket \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor - \mathbf{m}_\lambda \rrbracket \end{aligned} \quad (8.24)$$

which is the possible location of the right front of the fire k at time $\mathbf{a}_\lambda(t - \mathbf{v}_{\lambda, \pi})$,

$$\begin{aligned} & \llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t + \mathbf{v}_{\lambda, \pi} - T_k - \varepsilon_\lambda) \rfloor, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t + \mathbf{v}_{\lambda, \pi} - T_k + \varepsilon_\lambda) \rfloor \rrbracket \\ & \subset \llbracket \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor + \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi} \rrbracket \end{aligned} \quad (8.25)$$

which is the possible location of the right front of the fire k at time $\mathbf{a}_\lambda(t + \mathbf{v}_{\lambda,\pi})$.

For $k \in \{1, \dots, n\}$ and $t \geq T_k$ there also holds true that

$$\lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - \mathbf{e}_{\lambda,\pi} - T_k + \varepsilon_\lambda) \rfloor \leq \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor \quad (8.26)$$

and

$$\lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - 4\mathbf{v}_{\lambda,\pi} - T_k + \varepsilon_\lambda) \rfloor \leq \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor - \mathbf{m}_\lambda - 3\mathbf{k}_{\lambda,\pi}, \quad (8.27)$$

$$\lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t + 4\mathbf{v}_{\lambda,\pi} - T_k - \varepsilon_\lambda) \rfloor \geq \lfloor \mathbf{n}_\lambda X_k^+(t) \rfloor + \mathbf{m}_\lambda + 3\mathbf{k}_{\lambda,\pi}. \quad (8.28)$$

Very similar estimations of course hold for $X_k^-(t)$.

Finally, for all $i \in I_A^\lambda$ and all $k \in \{1, \dots, n\}$, there holds that

$$\left[T_k + \frac{|i - \lfloor \mathbf{n}_\lambda x \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_k + \frac{|i - \lfloor \mathbf{n}_\lambda x \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda \right] \subset \left[T_k \left(\frac{i}{\mathbf{n}_\lambda} \right) - \mathbf{e}_{\lambda,\pi}, T_k \left(\frac{i}{\mathbf{n}_\lambda} \right) + \mathbf{e}_{\lambda,\pi} \right] \quad (8.29)$$

which has to be seen as the time interval where a tree may be burn due to the fire k .

STAGE 0.

In this Stage we fix some $s_0 \in [0, T]$ and work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{s_0}^{\lambda,\pi}$. We deduce an estimate of the last time that a given site has been burned.

Lemma 8.5. *Let $s_0 \in [0, T]$ and q_0 such that $s_0 \in [T_{q_0}, T_{q_0+1})$. On $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{s_0}^{\lambda,\pi}$, for all $(i, t) \in I_A^\lambda \times [0, s_0]$ such that $i \notin \bigcup_{x \in \chi_t} \langle x \rangle_{\lambda,\pi} \cup \bigcup_{1 \leq k \leq q_0} ([X_k^{D,+}]_{\lambda,\pi} \cup [X_k^{D,-}]_{\lambda,\pi})$,*

1. $\tau_t(i/\mathbf{n}_\lambda) = 0$ if and only if $\rho_t^{\lambda,\pi}(i) = 0$;
2. if $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda)$, for some $k \in \{1, \dots, q_0\}$, then

$$\rho_t^{\lambda,\pi}(i) \in \left[T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda \right].$$

Observe that for (i, t) be as in the statement, in the two cases, there holds that, using (8.29),

$$\left| \rho_t^{\lambda,\pi}(i) - \tau_t(i/\mathbf{n}_\lambda) \right| \leq \mathbf{e}_{\lambda,\pi}.$$

For $t \in [0, s_0]$ and $x \in (-A, A)$ in such a way that $[x]_{\lambda,\pi} \cap [y]_{\lambda,\pi} = \emptyset$ for all $y \in \chi_t \cup \mathcal{B}_M^D$, if $\tau_t(x) = T_l(x)$, for some $l \in \{1, \dots, n\}$, then by construction $\tau_t(i/\mathbf{n}_\lambda) = T_l(i/\mathbf{n}_\lambda)$ for all $i \in [x]_{\lambda,\pi}$. Thus, using (8.24) and (8.25), Lemma 8.5 implies that for all $i \in (x)_\lambda$,

$$\left| \rho_t^{\lambda,\pi}(i) - \tau_t(x) \right| \leq \mathbf{v}_{\lambda,\pi}$$

whence, using (8.27) and (8.28), for all $i \in [x]_{\lambda,\pi}$, there holds that

$$\left| \rho_t^{\lambda,\pi}(i) - \tau_t(x) \right| \leq 4\mathbf{v}_{\lambda,\pi}.$$

Proof. Let $s_0 \in [0, T]$ and q_0 such that $s_0 \in [T_{q_0}, T_{q_0+1})$.

Step 1. The key of the proof is the observation that if a site $i \in I_A^\lambda$ is burning at time $\mathbf{a}_\lambda t \leq \mathbf{a}_\lambda s_0$ then there exists $k \in \{1, \dots, q_0\}$ such that $\zeta_{\mathbf{a}_\lambda(t-T_k)}^{\lambda,\pi,k}(i - \lfloor \mathbf{n}_\lambda X_k \rfloor) = 2$ (a burning tree in the (λ, π, A) -FFP corresponds to a burning tree in some propagation process).

Indeed, assume that a match falls on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k \leq \mathbf{a}_\lambda t$. Recall that the propagation process ignited at (X_k, T_k) is defined using the seed processes $(N_t^{S,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^{P,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$. Thus, with our coupling, the right front of the fire in the propagation process $(\zeta_t^{\lambda,\pi,k}(i))_{t \geq 0, i \in \mathbb{Z}}$ at some time $\mathbf{a}_\lambda s$ is $i_{\mathbf{a}_\lambda s}^{k,+}$ whence the (hypothetical) right front of the (λ, π, A) -FFP at time $\mathbf{a}_\lambda(s + T_k)$ is $\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda s}^{k,+}$. Recall that a spark in the propagation

process $(\zeta_t^{\lambda,\pi,k}(i))_{t \geq 0, i \in \mathbb{Z}}$ corresponds to a site $i \in \mathbb{Z}$ where a seed has fallen between the instant at which i propagates for the first time and the instant at which $i + 1$ if $i \geq 0$ or $i - 1$ if $i \leq 0$ propagates for the first time. On $\Omega_{\lambda,\pi}^{P,T}(X_k, T_k)$, such a spark has vacant neighbors. Thus, with our coupling, the site $\lfloor \mathbf{n}_\lambda X_k \rfloor + i$ is a spark in the (λ, π) -FFP (that is a burning tree which is not a front of a fire) if the site i is a spark in the propagation process. Such a spark in the (λ, π, A) -FFP has inevitably vacant neighbors.

Step 2. By Step 1, Lemma 4.2 and (8.23), we deduce that a burning tree at time $\mathbf{a}_\lambda t$ in the (λ, π, A) -FFP necessarily belongs to

$$\llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - T_k - \varepsilon_\lambda) \rfloor, \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(t - T_k + \varepsilon_\lambda) \rfloor \rrbracket \subset \langle X_k^+(t) \rangle_{\lambda,\pi}$$

or to

$$\llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor - \lfloor \mathbf{a}_\lambda \pi(t - T_k + \varepsilon_\lambda) \rfloor, \lfloor \mathbf{n}_\lambda X_k \rfloor - \lfloor \mathbf{a}_\lambda \pi(t - T_k - \varepsilon_\lambda) \rfloor \rrbracket \subset \langle X_k^-(t) \rangle_{\lambda,\pi}$$

for some $k \in \{1, \dots, q_0\}$ such that $T_k \leq t$.

Conversely, if a site $i \in I_A^\lambda$ is burning at time $\mathbf{a}_\lambda t \leq \mathbf{a}_\lambda s_0$ then there is $k \in \{1, \dots, n\}$ such that, recalling (8.29),

$$t \in \left[T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda \right] \subset \left(T_k \left(\frac{i}{\mathbf{n}_\lambda} \right) - \mathbf{e}_{\lambda,\pi}, T_k \left(\frac{i}{\mathbf{n}_\lambda} \right) + \mathbf{e}_{\lambda,\pi} \right).$$

Step 3. Next, we observe that if a site j is burning at some time $\mathbf{a}_\lambda u \leq \mathbf{a}_\lambda s_0$, then there is $k \in \{1, \dots, q_0\}$ such that $u \in [T_k + (T_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}^k / \mathbf{a}_\lambda), T_k + \frac{|j - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$ and for all $s \in [T_k, T_k + (T_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}^k / \mathbf{a}_\lambda)]$ we have

$$\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s-T_k)}^{k,+}) = 2$$

if $j \geq \lfloor \mathbf{n}_\lambda X_k \rfloor$ while if $j \leq \lfloor \mathbf{n}_\lambda X_k \rfloor$, we have

$$\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s-T_k)}^{k,-}) = 2.$$

Indeed, by construction, a fire starting on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$, for some $k \in \{1, \dots, q_0\}$, does not affect the site j before $\mathbf{a}_\lambda T_k + T_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}^k$ and by $\Omega_{\lambda,\pi}^{P,T}(X_k, T_k)$, as been checked on Step 1, does not affect the site j after $\mathbf{a}_\lambda T_k + \frac{|j - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\pi} + \mathbf{a}_\lambda \varepsilon_\lambda$.

Assume *e.g.* that $j \geq \lfloor \mathbf{n}_\lambda X_k \rfloor$ and that there is $s \in [T_k, T_k + (T_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}^k / \mathbf{a}_\lambda)]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s-T_k)}^{k,+}) = 0$: the right front reaches a vacant site. Since sparks has vacant neighbors, the right front can not propagate more and is stopped (after a while, thanks to our coupling). Hence, the right front cannot reach j .

Step 4. Here we prove that for i and t be as in the statement and if $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda) > 0$, for some $k \in \{1, \dots, q_0\}$, then i is not affected (in the discrete process) by any fire during the time interval $[\mathbf{a}_\lambda(T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda), \mathbf{a}_\lambda t]$.

Assume *e.g.* that $i/\mathbf{n}_\lambda = X_k^+(T_k(i/\mathbf{n}_\lambda)) \in \chi_{T_k(i/\mathbf{n}_\lambda)}^+$. We have $i/\mathbf{n}_\lambda \leq X_k^{D,+}$ and $T_k(i/\mathbf{n}_\lambda) \leq T_k^{D,+}$ whence $\lfloor \mathbf{n}_\lambda X_k \rfloor \leq i \leq \lfloor \mathbf{n}_\lambda X_k^{D,+} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}$ (because $i \notin [X_k^{D,+}]_{\lambda,\pi}$) and $T_k(i/\mathbf{n}_\lambda) + \mathbf{v}_{\lambda,\pi} \leq T_k^{D,+}$ (thanks to (8.24)).

So that there is $u_0 \in [T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda, t]$ such that the site i is burning at time $\mathbf{a}_\lambda u_0$, it is necessary that there is $l \neq k$ such that $u_0 \in [T_l + \frac{|i - \lfloor \mathbf{n}_\lambda X_l \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_l + \frac{|i - \lfloor \mathbf{n}_\lambda X_l \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$, recall Step 3, with

$$\eta_{\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l}^{\lambda,\pi}(j) = 2 \text{ for all } j \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor, i \rrbracket$$

if $i \geq \lfloor \mathbf{n}_\lambda X_k \rfloor$, or

$$\eta_{\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l}^{\lambda,\pi}(j) = 2 \text{ for all } j \in \llbracket i, \lfloor \mathbf{n}_\lambda X_l \rfloor \rrbracket$$

if $i \leq \lfloor \mathbf{n}_\lambda X_k \rfloor$.

If $i/\mathbf{n}_\lambda = X_l^+(T_l(i/\mathbf{n}_\lambda))$, then $i \geq \lfloor \mathbf{n}_\lambda X_l^{D,+} \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}$ whence $T_l(i/\mathbf{n}_\lambda) \geq T_l^{D,+} + \mathbf{v}_{\lambda,\pi}$, thanks to (8.25). Indeed

- (a) if $t \in [T_l + \frac{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_l + \frac{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$, then $i \in \langle X_l^+(t) \rangle_{\lambda, \pi}$. Since $i \notin \bigcup_{x \in \chi_t} \langle x \rangle_{\lambda, \pi}$, we deduce that $X_l^+(t) \notin \chi_t^+$ whence $T_l^{D,+} \leq t$. But $i \notin [X_l^{D,+}]_{\lambda, \pi}$, thus $T_l^{D,+} < t - \mathbf{v}_{\lambda, \pi}$, recall (8.25), and $i \geq \lfloor \mathbf{n}_\lambda X_l^{D,+} \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}$;
- (b) if $t \geq T_l + \frac{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda \geq T_k + \frac{i - \lfloor \mathbf{n}_\lambda X_k \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$ and $i \leq \lfloor \mathbf{n}_\lambda X_l^{D,+} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$, using $\Omega_t^{\lambda, \pi}$, we deduce that $T_l(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi} \leq t$ and $T_l(i/\mathbf{n}_\lambda) + \mathbf{v}_{\lambda, \pi} \leq T_l^{D,+}$, recall (8.23) and (8.24). Thus, $F_{T_l(i/\mathbf{n}_\lambda)}(i/\mathbf{n}_\lambda) = 1$. But by construction there holds that $|T_l(i/\mathbf{n}_\lambda) - T_k(i/\mathbf{n}_\lambda)| \geq 3\alpha$, thanks to $\Omega_M(\alpha)$, whence $T_l(i/\mathbf{n}_\lambda) \geq T_k(i/\mathbf{n}_\lambda) + 3\alpha$, a contradiction since $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda)$. Thus, $i \geq \lfloor \mathbf{n}_\lambda X_l^{D,+} \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}$, as desired.

If $i/\mathbf{n}_\lambda = X_l^-(T_l(i/\mathbf{n}_\lambda))$, then $i \leq \lfloor \mathbf{n}_\lambda X_l^{D,-} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$ whence $T_l(i/\mathbf{n}_\lambda) \geq T_l^{D,-} + \mathbf{v}_{\lambda, \pi}$, thanks to (8.25). Indeed

- (a') if $t \in [T_l + \frac{\lfloor \mathbf{n}_\lambda X_l \rfloor - i}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_l + \frac{\lfloor \mathbf{n}_\lambda X_l \rfloor - i}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$, we conclude as in case (a) above that $T_l^{D,-} \leq t - \mathbf{v}_{\lambda, \pi}$ and $i \leq \lfloor \mathbf{n}_\lambda X_l^{D,-} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$;
- (b') if $t \geq T_l + \frac{\lfloor \mathbf{n}_\lambda X_l \rfloor - i}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda \geq T_k + \frac{i - \lfloor \mathbf{n}_\lambda X_k \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda$ and $i \geq \lfloor \mathbf{n}_\lambda X_l^{D,-} \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}$, using $\Omega_t^{\lambda, \pi}$, we deduce that $T_l(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi} \leq t$ and $T_l(i/\mathbf{n}_\lambda) + \mathbf{v}_{\lambda, \pi} \leq T_l^{D,-}$, thanks to (8.23) and (8.24). Thus, $F_{T_l(i/\mathbf{n}_\lambda)}(i/\mathbf{n}_\lambda) = 1$ and $Z_{T_l(i/\mathbf{n}_\lambda)}(i/\mathbf{n}_\lambda) = 1$ whence $T_l(i/\mathbf{n}_\lambda) \geq T_k(i/\mathbf{n}_\lambda) + 1$, a contradiction since $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda)$. Thus $i \leq \lfloor \mathbf{n}_\lambda X_l^{D,-} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$, as desired.

Using $\Omega_t^{\lambda, \pi}$, we deduce that

- if $i/\mathbf{n}_\lambda = X_l^+(T_l(i/\mathbf{n}_\lambda))$, there is $s \in [T_l^{D,+} - \mathbf{v}_{\lambda, \pi}, T_l^{D,+} + \mathbf{v}_{\lambda, \pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,+}) = 0$ whence $\eta_{\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda, \pi}(j) = 0$ for some $j \in [X_l^{D,+}]_{\lambda, \pi}$, thanks to (8.24) and (8.25);
- if $i/\mathbf{n}_\lambda = X_l^-(T_l(i/\mathbf{n}_\lambda))$, there is $s \in [T_l^{D,-} - \mathbf{v}_{\lambda, \pi}, T_l^{D,-} + \mathbf{v}_{\lambda, \pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,-}) = 0$ whence $\eta_{\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda, \pi}(j) = 0$ for some $j \in [X_l^{D,-}]_{\lambda, \pi}$, thanks to (8.24) and (8.25).

Thus, the site i can not be burned during the time interval $[T_k + \frac{i - \lfloor \mathbf{n}_\lambda X_k \rfloor}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda, t]$.

Step 5. Here we prove that for i and t be as in the statement, if $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda) > 0$ for some $k \in \{1, \dots, n\}$, then $\eta_{\mathbf{a}_\lambda T_k + T_{i-\lfloor \mathbf{n}_\lambda X_k \rfloor}}^{\lambda, \pi}(i) = 2$.

Indeed, assume for example that $i/\mathbf{n}_\lambda = X_k^+(T_k(i/\mathbf{n}_\lambda))$, for some $k \in \{1, \dots, n\}$. By construction, there holds that $i/\mathbf{n}_\lambda \leq X_k^{D,+}$ and $i/\mathbf{n}_\lambda \leq X_k^+(s_0)$ whence $\lfloor \mathbf{n}_\lambda X_k \rfloor \leq i \leq \lfloor \mathbf{n}_\lambda X_k^{D,+} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$ (because $i \notin [X_k^{D,+}]_{\lambda, \pi}$) and $\lfloor \mathbf{n}_\lambda X_k \rfloor \leq i \leq \lfloor \mathbf{n}_\lambda X_k^+(s_0) \rfloor - \mathbf{k}_{\lambda, \pi}$ (because if $s_0 \leq T_k^{D,+}$ then $i \notin \langle X_k^+(s_0) \rangle_{\lambda, \pi}$ and if $s_0 > T_k^{D,+}$ then $\lfloor \mathbf{n}_\lambda X_k^+(s_0) \rfloor - \mathbf{k}_{\lambda, \pi} \geq \lfloor \mathbf{n}_\lambda X_k^{D,+} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$). We distinguish two cases.

- If $s_0 \geq T_k^{D,+} - \mathbf{v}_{\lambda, \pi}$, then by $\Omega_{s_0}^{\lambda, \pi}$, we deduce that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s-T_k)}^{k,+}) = 2$ for all $s \in [T_k, T_k^{D,+} - \mathbf{v}_{\lambda, \pi}]$. This also implies, thanks to (8.24), that $\eta_{\mathbf{a}_\lambda T_k + T_{j-\lfloor \mathbf{n}_\lambda X_k \rfloor}}^{\lambda, \pi}(j) = 2$ for all $j \in [\lfloor \mathbf{n}_\lambda X_k \rfloor, \lfloor \mathbf{n}_\lambda X_k^{D,+} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}]$. It especially holds for i , thanks to the previous observation.
- If $s_0 < T_k^{D,+} - \mathbf{v}_{\lambda, \pi}$, we deduce, by $\Omega^{P,T}(\lambda, \pi)$, (8.23) and the previous observation, that

$$\lfloor \mathbf{n}_\lambda X_k \rfloor \leq i \leq \lfloor \mathbf{n}_\lambda X_k^+(s_0) \rfloor - \mathbf{k}_{\lambda, \pi} \leq \lfloor \mathbf{n}_\lambda X_k \rfloor + \lfloor \mathbf{a}_\lambda \pi(s_0 - T_k - \varepsilon_\lambda) \rfloor \leq \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s_0 - T_k)}^{k,+}. \quad (8.30)$$

Finally, by $\Omega_{s_0}^{\lambda, \pi}$, we have $\eta_{\mathbf{a}_\lambda u}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(u-T_k)}^{k,+}) = 2$ for all $u \in [T_k, s_0]$ which implies the claim.

Step 6. We now conclude in the case $\tau_t(i/\mathbf{n}_\lambda) = T_k(i/\mathbf{n}_\lambda) > 0$. By Step 4, we deduce that

$$\rho_t^{\lambda,\pi}(i) \leq T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda.$$

By Step 5, we deduce that $\rho_t^{\lambda,\pi}(i) \geq T_k + T_{i - \lfloor \mathbf{n}_\lambda X_k \rfloor}^k / \mathbf{a}_\lambda$ and conclude using $\Omega^{P,T}(\lambda, \pi)$ that

$$\rho_t^{\lambda,\pi}(i) \geq T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda.$$

Step 7. Finally, if $\tau_t(i/\mathbf{n}_\lambda) = 0$, we conclude, using similar argument as in Step 4 (recall that $i \notin \bigcup_{1 \leq k \leq q_0} ([X_k^{D,+}]_{\lambda,\pi} \cup [X_k^{D,-}]_{\lambda,\pi})$), that no fire can affect the site i until $\mathbf{a}_\lambda t$ and thus $\rho_t^{\lambda,\pi}(i) = 0$.

Conversely, if $\rho_t^{\lambda,\pi}(i) = 0$, then for all $l \in \{1, \dots, n\}$ such that $T_l(i/\mathbf{n}_\lambda) < t$, we necessarily have $F_{T_l(i/\mathbf{n}_\lambda)}(i/\mathbf{n}_\lambda) = 0$ (else, applying $\Omega_t^{\lambda,\pi}$, one should have $\eta_{\mathbf{a}_\lambda T_l + T_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda,\pi}(i) = 2$). This concludes the proof. \square

STAGE 1.

The aim of this stage is to prove that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q}^{\lambda,\pi}$ implies $\Omega_{T_q + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$.

Observe that for all $i \in I_A^\lambda \setminus \{\lfloor \mathbf{n}_\lambda X_q \rfloor\}$,

$$\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(i) = \eta_{\mathbf{a}_\lambda T_q -}^{\lambda,\pi}(i)$$

while

$$\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 2\mathbf{1}_{\{\eta_{\mathbf{a}_\lambda T_q -}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 1\}}.$$

First, we situate the burning trees at time $\mathbf{a}_\lambda T_q$ for the (λ, π, A) -FFP.

Lemma 8.6. *We work on $\Omega_{T_q}^{\lambda,\pi} \cap \Omega(\alpha, \gamma, \lambda, \pi)$.*

1. *At time $\mathbf{a}_\lambda T_q$, a burning tree which is not located at $\lfloor \mathbf{n}_\lambda X_q \rfloor$ necessarily belongs to $\langle x \rangle_{\lambda,\pi}$, for some $x \in \chi_{T_q}^+ \cup \chi_{T_q}^- \subset \mathcal{B}_{M,q}^1$, and is either at $\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,+}$ or at $\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,-}$, for some $k < q$, or has vacant neighbors.*
2. *If $X_k^+(T_q) = X_k + \frac{T_q - T_k}{p} \in \chi_{T_q}^+$ for some $k < q$, then $\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,+}) = 2$ and $\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(i) = 1$ for all $i \in [\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,+} + 1, \lfloor \mathbf{n}_\lambda(X_k + 2\alpha/p) \rfloor]$.*
3. *If $X_k^-(T_q) = X_k - \frac{T_q - T_k}{p} \in \chi_{T_q}^-$ for some $k < q$, then $\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,-}) = 2$ and $\eta_{\mathbf{a}_\lambda T_q}^{\lambda,\pi}(i) = 1$ for all $i \in [\lfloor \mathbf{n}_\lambda(X_k - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,-} - 1]$.*

Proof. First, observe that, by $\Omega_M(\alpha)$, $|x - y| > 3\alpha/p$ for all $x, y \in \mathcal{B}_{M,q}^1 \cup \mathcal{B}_M^D$ with $x \neq y$. Hence, for all $x \in \mathcal{B}_{M,q}^1$, there is a unique $k < q$ such that $x = X_k^+(T_q)$ or $x = X_k^-(T_q)$.

In the whole proof, we work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q}^{\lambda,\pi}$.

Step 1. We first prove 1. As claimed in Step 2 in the proof of Lemma 8.5, due to $\Omega^{P,T}(\lambda, \pi)$, if a tree burns at time $\mathbf{a}_\lambda T_q$ in the (λ, π, A) -FFP, it necessarily belongs to $\langle X_k^+(T_q) \rangle_{\lambda,\pi}$ or $\langle X_k^-(T_q) \rangle_{\lambda,\pi}$ for some $k < q$ and is either $\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,+}$ or $\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,-}$, or has vacant neighbors.

It remains to prove that if $x \in \mathcal{B}_{M,q}^1 \setminus (\chi_{T_q}^+ \cup \chi_{T_q}^-)$, then there is no burning tree in $\langle x \rangle_{\lambda,\pi}$ at time $\mathbf{a}_\lambda T_q$. We assume *e.g.* that $x = X_k^+(T_q)$ for some $k < q$. Since $x \notin \chi_{T_q}^+$, there holds that $T_k^{D,+} \leq T_q$ whence $T_k^{D,+} \leq T_q - 3\alpha$ and $x \geq X_k^{D,+} + 3\alpha/p$, due to $\Omega_M(\alpha)$. We deduce, by $\Omega_{T_q}^{\lambda,\pi}$, that there is $s \in [T_k^{D,+} - \mathbf{v}_{\lambda,\pi}, T_k^{D,+} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(s - T_k)}^{k,+}) = 0$ whence as usual (using (8.24) and (8.25)) that there is $j \in [X_k^{D,+}]_{\lambda,\pi}$ such that $\eta_{\mathbf{a}_\lambda T_k + T_{j - \lfloor \mathbf{n}_\lambda X_k \rfloor}}^{\lambda,\pi}(j) = 0$. Since k is unique, we conclude, using same arguments as in Step 3 in the proof of Lemma 8.5, that there can not be

burning tree in $\langle x \rangle_{\lambda, \pi}$ at time $\mathbf{a}_\lambda T_q$ (because the right front has been stopped in $[X_k^{D,+}]_{\lambda, \pi}$ and $\lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{k}_{\lambda, \pi} \geq \lfloor X_k^{D,+} \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}$).

Step 2. We next prove 2. Let $k < q$. We set $x := X_k^+(T_q) \in \mathcal{B}_{M,q}^1$. Since $x \notin \mathcal{B}_m^D$, we have $T_k^{D,+} > T_q > T_k$ whence, by $\Omega_M(\alpha)$, $T_k^{D,+} > T_q + 3\alpha > T_k + 6\alpha$. Recall that, since $Z_{T_q-}(x) = 1$, there holds that $T_q - \tau_{T_q-}(x) \geq 1$ whence $T_q - \tau_{T_q-}(x) \geq 1 + 3\alpha$, thanks to $\Omega_M(\alpha)$. We deduce that $Z_{T_q-}(y) = 1$ and $T_q - \tau_{T_q-}(y) \geq 1 + \alpha$ for all $y \in [x, x + 2\alpha/p]$. We set $\tau_{T_q-}(x) = T_l(x)$, for some $l \in \{0, \dots, q-1\}$.

Let us fix $i \in \llbracket \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{k}_{\lambda, \pi} + 1, \lfloor \mathbf{n}_\lambda(x + 2\alpha/p) \rfloor \rrbracket$. Observing that $i \notin \bigcup_{x \in \chi_{T_q}} \langle x \rangle_{\lambda, \pi} \cup \bigcup_{1 \leq k \leq q} ([X_k^{D,+}]_{\lambda, \pi} \cup [X_k^{D,-}]_{\lambda, \pi})$, we deduce from Lemma 8.5 and by (8.29) that $\rho_{T_q-}^{\lambda, \pi}(i) \leq \tau_{T_q-}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}$ whence

$$\rho_{T_q-}^{\lambda, \pi}(i) \leq T_q - 1 - \alpha + \mathbf{e}_{\lambda, \pi}.$$

We conclude using $\Omega_3^S(\lambda, \pi)$ that i is occupied at time $\mathbf{a}_\lambda T_q$.

Let now $i \in \llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k,+} + 1, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{k}_{\lambda, \pi} \rrbracket$. The site i has not (yet) been affected by the fire k . Observe that if $\rho_{T_q-}^{\lambda, \pi}(i) = 0$, since $T_q \geq 1$, we deduce by $\Omega_3^S(\lambda, \pi)$ that i is occupied at time $\mathbf{a}_\lambda T_q$. If $\rho_{T_q-}^{\lambda, \pi}(i) > 0$, by $\Omega^{P,T}(\lambda, \pi)$, we necessarily have $\rho_{T_q-}^{\lambda, \pi}(i) \in [T_l + \frac{|i - \lfloor \mathbf{n}_\lambda X_l \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_l + \frac{|i - \lfloor \mathbf{n}_\lambda X_l \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda]$. We deduce as above that

$$\rho_{T_q-}^{\lambda, \pi}(i) \leq T_l(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi} \leq T_q - 1 - \alpha + \mathbf{e}_{\lambda, \pi}$$

and conclude using $\Omega_3^S(\lambda, \pi)$ that i is occupied at time $\mathbf{a}_\lambda T_q$.

Step 3. Finally, point 3 is proved exactly as Point 2. \square

We finally examine the (λ, π, A) -FFP around $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $\mathbf{a}_\lambda T_q$.

Lemma 8.7. *We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q}^{\lambda, \pi}$.*

1. *If $Z_{T_q-}(X_q) < 1$ then there are $j_1, j_2 \in (X_q)_\lambda$ such that $j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$ and $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(j_1) = \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(j_2) = 0$ for all $s \in [T_q, T_q + \kappa_{\lambda, \pi}^0]$.*
2. *If $Z_{T_q-}(X_q) = 1$ then $\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) = 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p) \rfloor \rrbracket$.*

Proof. First observe that $|x - X_q| > 3\alpha/p$ for all $y \in \mathcal{B}_{M,q}^1 \cup \mathcal{B}_m^D$ whence $F_{T_q-}(y) = 0$ for all $y \in (X_q - 3\alpha/p, X_q + 3\alpha/p)$. We deduce, by Lemma 8.6, that there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p) \rfloor \rrbracket$ at time $\mathbf{a}_\lambda T_q-$ in the (λ, π, A) -FFP. We distinguish two cases.

Step 1. We first study the case $\tau_{T_q-}(X_q) > 0$. By construction, recalling (8.17) and since no match has fallen in X_q during $[0, T_q)$, there is a unique $k < q$ such that $\tau_{T_q-}(y) = T_k(y)$ for all $y \in (X_q - 3\alpha/p, X_q + 3\alpha/p)$.

If $Z_{T_q-}(X_q) < 1$, then $Z_{T_q-}(X_q) = T_q - \tau_{T_q-}(X_q) < 1$ whence $T_q - \tau_{T_q-}(X_q) < 1 - 3\alpha$, thanks to $\Omega_M(\alpha)$. Recall that for $i \in (X_q)_\lambda$, seeds fall according to $(N_t^{S,q}(i - \lfloor \mathbf{n}_\lambda X_q \rfloor))_{t \geq 0}$.

By Lemma 8.5, for all $i \in (X_q)_\lambda$,

$$\begin{aligned} \rho_{T_q-}^{\lambda, \pi}(i) &\in [T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} - \varepsilon_\lambda, T_k + \frac{|i - \lfloor \mathbf{n}_\lambda X_k \rfloor|}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda] \\ &\subset (\tau_{T_q-}(X_q) - \mathbf{v}_{\lambda, \pi}, \tau_{T_q-}(X_q) + \mathbf{v}_{\lambda, \pi}). \end{aligned}$$

Since we work on $\Omega_2^S(\lambda, \pi)$ and since $T_q, \tau_{T_q-}(X_q) \in \mathcal{B}_M \cup \mathcal{B}_{M,q}^1$, there are some $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that no seed has fallen on i_1 and on i_2 during $[\mathbf{a}_\lambda(\tau_{T_q-}(X_q) - 4\mathbf{v}_{\lambda, \pi}), \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi})] \supset [\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^0)]$. All this implies that i_1 and i_2 remain vacant during (at least) the time interval $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^0)]$.

If $Z_{T_q-}(X_q) = 1$, then $T_q - \tau_{T_q-}(X_q) \geq 1$ whence $T_q - \tau_{T_q-}(X_q) > 1 + 3\alpha$ and $T_q - \tau_{T_q-}(y) > 1 + \alpha$ for all $y \in (x - 2\alpha/p, x + 2\alpha/p)$, thanks to $\Omega_M(\alpha)$.

By Lemma 8.5, for all $i \in \llbracket \lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p) \rfloor \rrbracket$, we deduce

$$\rho_{T_q-}^{\lambda, \pi}(i) \in [T_k(i/\mathbf{n}_\lambda) - \mathbf{e}_{\lambda, \pi}, T_k(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}].$$

Since we work on $\Omega_3^S(\lambda, \pi)$, at least one seed has fallen on each site during $[\mathbf{a}_\lambda(T_k(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}), \mathbf{a}_\lambda(T_k(i/\mathbf{n}_\lambda) + 1 + \mathbf{e}_{\lambda, \pi})] \subset [\mathbf{a}_\lambda(T_k(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}), \mathbf{a}_\lambda T_q]$. Since, by definition, i cannot be affected by a fire during $(\rho_{T_q-}^{\lambda, \pi}(i), \mathbf{a}_\lambda T_q)$, we deduce that the zone $\llbracket \lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p) \rfloor \rrbracket$ is completely filled at time $\mathbf{a}_\lambda T_q$.

Step 2. Here we study the case $\tau_{T_q-}(X_q) = 0$. By $\Omega_M(\alpha)$, we have $\tau_{T_q-}(y) = 0$ for all $y \in (X_q - 3\alpha/p, X_q + 3\alpha/p)$.

If $Z_{T_q-}(X_q) < 1$, then $Z_{T_q}(X_q) = T_q < 1$ whence $T_q < 1 - 3\alpha$. Since we still work on $\Omega_2^S(\lambda, \pi)$, there are some $-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$ such that no seed has fallen on i_1 and on i_2 during $[0, \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi})] \supset [0, \mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^0)]$. Since we start with a vacant initial configuration, we deduce that i_1 and i_2 remain vacant during (at least) the time interval $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \kappa_{\lambda, \pi}^0)]$.

If $Z_{T_q-}(X_q) = 1$, then $T_q > 1$ whence $T_q > 1 + 3\alpha$. By Lemma 8.5 we deduce that $\rho_{T_q-}^{\lambda, \pi}(i) = 0$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p) \rfloor \rrbracket$ and thus

$$\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) = \min(N_{\mathbf{a}_\lambda T_q-}^{S, \lambda, \pi}(i), 1).$$

Since we work on $\Omega_3^S(\lambda, \pi)$, at least one seed has fallen on each site during $[0, \mathbf{a}_\lambda] \subset [0, \mathbf{a}_\lambda T_q]$. All this implies that the zone $\llbracket \lfloor \mathbf{n}_\lambda(X_q - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(X_q + 2\alpha/p) \rfloor \rrbracket$ is completely filled at time $\mathbf{a}_\lambda T_q$. \square

The following corollary completes Stage 1.

Corollary 8.8. *On $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_q + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$.*

Proof. Let $k < q$ such that $T_k^{D, +} \in (T_q, T_{q+1})$. By $\Omega_M(\alpha)$, we have $T_q + 3\alpha < T_k^{D, +}$ whence $T_q + 4\mathbf{v}_{\lambda, \pi} < T_k^{D, +} - \mathbf{v}_{\lambda, \pi}$. Thus, no fire extinguishes during $[T_q, T_q + 4\mathbf{v}_{\lambda, \pi}]$ (in the limit process). Hence, we have to prove that

- if $X_k^+(T_q) \in \chi_{T_q}^+$, for some $k \leq q$, then $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(t-T_k)}^{k, +}) = 2$ for all $t \in [T_q, T_q + 4\mathbf{v}_{\lambda, \pi}]$;
- if $X_k^-(T_q) \in \chi_{T_q}^-$, for some $k \leq q$, then $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(t-T_k)}^{k, -}) = 2$ for all $t \in [T_q, T_q + 4\mathbf{v}_{\lambda, \pi}]$;
- if $Z_{T_q-}(X_q) < 1$, then the left and right fronts of the fire ignited at (X_q, T_q) are stopped during the time interval $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \mathbf{v}_{\lambda, \pi})]$.

Observe that, on $\Omega^{P, T}(\lambda, \pi)$ there a.s. holds that, for all $k \leq q$,

$$0 \leq i_{\mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi} - T_k)}^{k, +} - i_{\mathbf{a}_\lambda(T_q - T_k)}^{k, +} \leq 4(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}) \leq \lfloor \mathbf{n}_\lambda \alpha / p \rfloor$$

and

$$-\lfloor \mathbf{n}_\lambda \alpha / p \rfloor \leq -4(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}) \leq i_{\mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi} - T_k)}^{k, -} - i_{\mathbf{a}_\lambda(T_q - T_k)}^{k, -} \leq 0.$$

All this implies that a front of a fire at time $\mathbf{a}_\lambda T_q$, which belong to $\langle x \rangle_{\lambda, \pi}$ for some $x \in \mathcal{B}_{M, q}^1 \cup \{\mathbf{n}_\lambda X_q\}$, can not affect the zone outside $\llbracket \lfloor \mathbf{n}_\lambda(x - \alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(x + \alpha/p) \rfloor \rrbracket$ during the time interval $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi})]$.

Step 1. Here we prove that for $k \leq q$ such that $x := X_k^+(T_q) \in \chi_{T_q}^+$ then $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(t-T_k)}^{k, +}) = 2$ for all $t \in [T_q, T_q + 4\mathbf{v}_{\lambda, \pi}]$.

Indeed, by Lemma 8.6-2 if $k < q$ or by Lemma 8.7-2 if $k = q$, there holds that

$$\eta_{\mathbf{a}_\lambda T_q}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k, +}) = 2$$

and

$$\eta_{\mathbf{a}_\lambda T_q}^{\lambda, \pi}(i) = 1 \text{ for all } i \in \llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k, +} + 1, \lfloor \mathbf{n}_\lambda(x + 2\alpha/p) \rfloor \rrbracket.$$

But by the previous consideration, no fire, except this one, can affect the zone $\llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(T_q - T_k)}^{k, +} + 1, \lfloor \mathbf{n}_\lambda(x + \alpha/p) \rfloor \rrbracket$ during $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi})]$ and conversely, this fire can not affect the zone outside $\llbracket \lfloor \mathbf{n}_\lambda(x - \alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda(x + \alpha/p) \rfloor \rrbracket$. Hence, the right front of the fire k is not stopped during the time interval $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + 4\mathbf{v}_{\lambda, \pi})]$, as desired.

Step 2. Let $k \leq q$, if $x := X_k^-(T_q) \in \chi_{T_q}^-$ then $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_k \rfloor + i_{\mathbf{a}_\lambda(t - T_k)}^{k, -}) = 2$ for all $t \in [T_q, T_q + 4\mathbf{v}_{\lambda, \pi}]$. This can be shown using similar arguments as in Step 1 above.

Step 3. If $Z_{T_q^-}(X_q) < 1$, we have $T_q = T_q^{D, +} = T_q^{D, -}$. By Lemma 8.7-1, we deduce that there are $j_1, j_2 \in (X_q)_\lambda$ such that $j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$ and

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(j_1) = \eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(j_2) = 0 \text{ for all } s \in [T_q, T_q + \kappa_{\lambda, \pi}^0].$$

Hence, on $\Omega_{\lambda, \pi}^{P, T}(X_q, T_q)$, $\eta_{\mathbf{a}_\lambda T_q + T_{j_1 - \lfloor \mathbf{n}_\lambda X_q \rfloor}^q}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{T_{j_1 - \lfloor \mathbf{n}_\lambda X_q \rfloor}^q}^{q, -}) = 0$ because $T_q + T_{j_1 - \lfloor \mathbf{n}_\lambda X_q \rfloor}^q / \mathbf{a}_\lambda \leq T_q + \kappa_{\lambda, \pi}^0$ and $\eta_{\mathbf{a}_\lambda T_q + T_{j_2 - \lfloor \mathbf{n}_\lambda X_q \rfloor}^q}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_{T_{j_2 - \lfloor \mathbf{n}_\lambda X_q \rfloor}^q}^{q, +}) = 0$ because $T_q + T_{j_2 - \lfloor \mathbf{n}_\lambda X_q \rfloor}^q / \mathbf{a}_\lambda \leq T_q + \kappa_{\lambda, \pi}^0$, as desired. \square

STAGE 2.

In this Stage, we assume that $\mathcal{A}_q \neq \emptyset$ and we fix $k \in \llbracket 0, N_q - 1 \rrbracket$. We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$ and prove that $\Omega_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$ a.s. holds. We repeatedly use the fact that no match falls in $[-A, A]$ during the time interval $[T_q^k + 4\mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \alpha]$. Observe that, for all $i \in I_A^\lambda$,

$$\eta_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})}^{\lambda, \pi}(i) = \eta_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})}^{\lambda, \pi}(i).$$

We first examine the position of the burning trees of the (λ, π, A) -FFP at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})$.

Lemma 8.9. *We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$.*

1. *At time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})$, a burning tree necessarily belongs to $\langle x \rangle_{\lambda, \pi}$, for some $x \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+ \cup \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^-$, and is either $\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, +}$ or $\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, -}$, for some $l \leq q$, or has vacant neighbors.*
2. *If $X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ for some $l \leq q$, then $\eta_{\mathbf{a}_\lambda T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, +}) = 2$ and $\eta_{\mathbf{a}_\lambda T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(i) = 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, +} + 1, \lfloor \mathbf{n}_\lambda(X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) + 2\alpha/p) \rfloor \rrbracket$.*
3. *If $X_l^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^-$ for some $l \leq q$, then $\eta_{\mathbf{a}_\lambda T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, -}) = 2$ and $\eta_{\mathbf{a}_\lambda T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(i) = 1$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda(X_l^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, -} - 1 \rrbracket$.*

Proof. The proof is very similar to the proof of Lemma 8.6.

Indeed, we prove point 1 using $\Omega^{P, T}(\lambda, \pi)$ (as in the proof of Lemma 8.5) which implies that a burning tree necessarily belongs to $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \rangle_{\lambda, \pi}$ or $\langle X_l^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \rangle_{\lambda, \pi}$ for some $l \leq q$ and is either $\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, +}$ or $\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_l)}^{l, -}$ or has vacant neighbors. Furthermore, if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) < X_{l'}^-(T_q^k + 4\mathbf{v}_{\lambda, \pi})$, for some $l \neq l'$, we deduce, by $\Omega_M(\alpha)$, that

$$X_{l'}^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) - X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) > (3\alpha - 8\mathbf{v}_{\lambda, \pi})/p > \frac{5\alpha}{2p}.$$

Thus, as claimed in Step 3 in the proof of Lemma 8.5, for a site i_0 in $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \rangle_{\lambda, \pi}$ is burning at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})$, since l is unique, it is necessary that

$$\eta_{\mathbf{a}_\lambda T_l + T_{j - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda, \pi}(j) = 2 \text{ for all } j \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor, i_0 \rrbracket.$$

But, if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \notin \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+$ then $T_l^{D,+} \leq T_q^k$. By $\Omega_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$, we deduce that there is $j \in [X_l^{D,+}]_{\lambda,\pi}$ such that $\eta_{\mathbf{a}_\lambda T_l + T_j^l - \lfloor \mathbf{n}_\lambda X_l \rfloor}^{\lambda,\pi}(j) = 0$ (because there is $s \in [T_l^{D,+} - \mathbf{v}_{\lambda,\pi}, T_l^{D,+} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,+}) = 0$, recall (8.24) and (8.25)). Since $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rangle_{\lambda,\pi} \cap [X_l^{D,+}]_{\lambda,\pi} = \emptyset$, thanks (8.28) (recall that $X_l^{D,+} = X_l^+(T_l^{D,+})$), there is no burning tree in $\langle X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rangle_{\lambda,\pi}$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$.

Point 2 (or point 3) is proved as in Lemma 8.6. Indeed if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+$, then $T_l^{D,+} \geq T_q^{k+1} \geq T_q^k + 3\alpha$ and $|X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) - y| > 2\alpha$ for all $y \in \mathcal{B}_M^D$. Furthermore, on $\Omega_M(\alpha)$, by construction, we have

$$\tilde{H}_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0 \text{ for all } y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) + (3\alpha - 4\mathbf{v}_{\lambda,\pi})/p)$$

Thus, we prove that $\eta_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(j) = 1$ for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda(X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) + 2\alpha/p) \rfloor \rrbracket$ by distinguishing the cases $j \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rfloor + \mathbf{k}_\lambda, \pi \rrbracket$ and $j \in \llbracket \lfloor \mathbf{n}_\lambda X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rfloor + \mathbf{k}_\lambda, \pi, \lfloor \mathbf{n}_\lambda(X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) + 2\alpha/p) \rfloor \rrbracket$ (recalling that $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \notin \mathcal{B}_M^D$). \square

We then compute the cluster destroyed by a microscopic fire. We use the notation introduced in Lemma 8.2.

Lemma 8.10. *Let $m \leq q$, if $Z_{T_m-}(X_m) < 1$, we define $t_0 = T_m - Z_{T_m-}(X_m)$, which is nothing but $\tau_{T_m-}(X_m)$, recall (8.17). We then define, recall (8.12) and (8.13),*

(i) if $t_0 = T_l(X_m) > 0$ for some $l < m$ and if $X_m = X_l^+(t_0)$,

$$\mathcal{M} := (\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} - \lfloor \mathbf{n}_\lambda X_m \rfloor; t_0, T_m);$$

(ii) if $t_0 = T_l(X_m) > 0$ for some $l < m$ and if $X_m = X_l^-(t_0)$,

$$\mathcal{M} := (\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{l,-} - \lfloor \mathbf{n}_\lambda X_m \rfloor; t_0, T_m);$$

(iii) if $t_0 = 0$,

$$\mathcal{M} := (0; 0, T_m),$$

Then, working on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$, in each case, there holds that

$$(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i))_{t \in [t_0 - \mathbf{v}_{\lambda,\pi}, T_m + \kappa_{\lambda,\pi}^0], i \in (X_m)_\lambda} = (\zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi, \mathcal{M}, m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t \in [t_0 - \mathbf{v}_{\lambda,\pi}, T_m + \kappa_{\lambda,\pi}^0], i \in (X_m)_\lambda}$$

where the last process is defined as in Lemma 8.2 using the seed processes family $(N_t^{S,m}(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^{P,m}(i))_{t \geq 0, i \in \mathbb{Z}}$.

This in particular implies that, still on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$,

$$C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_m, T_m)) = \llbracket \lfloor \mathbf{n}_\lambda X_m \rfloor + i^g, \lfloor \mathbf{n}_\lambda X_m \rfloor + i^d \rrbracket \subset (X_m)_\lambda$$

where $\llbracket i^g, i^d \rrbracket = C^P((\zeta_t^{\lambda,\pi, \mathcal{M}, m}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, T_m)) \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$, recall Lemma 8.2.

Proof. We only treat the case (i). The case (ii) is of course similar and the case (iii) is easier.

We thus fix $1 \leq l < m \leq q$ in such a way that

$$\tau_{T_m-}(X_m) = t_0 = T_l(X_m) \text{ and } X_m = X_l^+(t_0).$$

By $\Omega_M(\alpha)$, we deduce that $T_l^{D,+} > t_0 + 3\alpha$ and $T_m > t_0 + 3\alpha > T_l + 6\alpha$. Hence, by construction, there holds that $Z_{t_0 - \mathbf{v}_{\lambda,\pi}}(y) = 1$ for all $y \in (X_m - \mathbf{v}_{\lambda,\pi}/p, X_m + 2\alpha/p)$. Observe that $T_q^k + 4\mathbf{v}_{\lambda,\pi} \geq T_m + \kappa_{\lambda,\pi}^0$.

By $\Omega_{T_q^k+4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$, we deduce that at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$ the site

$$\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} \in \llbracket \lfloor \mathbf{n}_\lambda X_m \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, \lfloor \mathbf{n}_\lambda X_m \rfloor - \mathbf{m}_\lambda \rrbracket$$

is burning whereas the zone $\llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda(X_m + 2\alpha/p) \rfloor \rrbracket$ is completely occupied (use very similar arguments as in Lemma 8.9-2, recalling that no match falls on X_m during $[0, T_m] \supset [0, t_0]$). Comparing $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t \geq 0, i \in \mathbb{Z}}$, we deduce that they are equal on $\llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{k,+}, \lfloor \mathbf{n}_\lambda X_m \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket \supset (X_m)_\lambda$ at time $\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi})$.

Since, with our coupling, seeds fall according to the same processes and fires spread according to the same processes on $[X_m]_{\lambda,\pi}$, we deduce that the fire preads in the same way through $\llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{k,+}, \lfloor \mathbf{n}_\lambda X_m \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket$. Thus, $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ remain equal on $\llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi} - T_l)}^{k,+} + 1, \lfloor \mathbf{n}_\lambda X_m \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} \rrbracket \supset (X_m)_\lambda$ during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi})]$, recall (8.28). No other fire affect the zone $(X_m)_\lambda$ until a match falls on $\lfloor \mathbf{n}_\lambda X_m \rfloor$ at time $\mathbf{a}_\lambda T_m$ because the zone $(X_m)_\lambda$ is protected by vacant site during the time interval $[\mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0)]$ (by construction for $\zeta^{\lambda,\pi,\mathcal{M},m}$ and because in the (λ, π, A) -FFP, on $\Omega_2^S(\lambda, \pi)$, there are

$$-\mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} < i_1 < -\mathbf{m}_\lambda < \mathbf{m}_\lambda < i_2 < \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}$$

where no seed fall during the time interval $(\mathbf{a}_\lambda(t_0 - 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0))$ and because the sites $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_1$ and $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_2$ has been made vacant by the fire l during $(\mathbf{a}_\lambda(t_0 - 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi}))$, recall (8.27) and (8.28)). Thus, since seeds fall on $[X_m]_{\lambda,\pi}$ according to the same processes, $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ remain equal on $(X_m)_\lambda$ during $[\mathbf{a}_\lambda(t_0 + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda T_m]$. Finally, by $\Omega_2^S(\lambda, \pi)$, we deduce that there are some sites

$$-\mathbf{m}_\lambda < i_3 < 0 < i_4 < \mathbf{m}_\lambda$$

where no seed fall during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0)]$ whence, as usual, in both cases, the sites $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_3$ and $\lfloor \mathbf{n}_\lambda X_m \rfloor + i_4$ are vacant during $[\mathbf{a}_\lambda(t_0 + \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0)]$, recall (8.25) (because they are made vacant by the fire l). Since the two processes evolve according to the same rules, the match falling on $\lfloor \mathbf{n}_\lambda X_m \rfloor$ at time $\mathbf{a}_\lambda T_m$ destroys the same zone. Thus, $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M},m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ are also equal on $(X_m)_\lambda$ during $[\mathbf{a}_\lambda T_m, \mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0)]$.

We deduce, on $\Omega_2^S(\lambda, \pi)$, as seen in **Micro**(p) in Subsection 4.4, that

$$C^P((\zeta_t^{\lambda,\pi,\mathcal{M},m}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, T_m)) := \llbracket i^g, i^d \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$$

and that there is no more burning tree in $(X_m)_\lambda$ at time $\mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0)$, whence

$$C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_m, T_m)) = \llbracket \lfloor \mathbf{n}_\lambda X_m \rfloor + i^g, \lfloor \mathbf{n}_\lambda X_m \rfloor + i^d \rrbracket \subset (X_m)_\lambda. \quad \square$$

We will need the following lemma.

Lemma 8.11. *Let $s_0 \in [T_q^k + \alpha, T_q^{k+1} + \alpha]$. We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^k+4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$.*

1. *In the limit process, if, for some $l \leq q$, $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k+4\mathbf{v}_{\lambda,\pi}}^+$ in such a way that $s_0 \leq T_l^{D,+}$ and*

$$F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0 \text{ for all } y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0 + \alpha)), \quad (8.31)$$

then, in the discrete process, the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ is not affected by a fire during the time interval $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})]$.

2. *In the limit process, if, for some $l \leq q$, $X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k+4\mathbf{v}_{\lambda,\pi}}^-$ in such a way that $s_0 \leq T_l^{D,-}$ and $F_{T_q^k+4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_l^-(s_0 + \alpha), X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}))$, then, in the discrete process, the site $\lfloor \mathbf{n}_\lambda X_l^-(s_0) \rfloor$ is not affected by a fire during the time interval $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})]$.*

Proof. It of course suffices to prove 1.

First, using (8.31), we deduce that

$$(X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0 + \alpha)) \cap (\chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+ \cup \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^-) = \emptyset.$$

Hence, by Lemma 8.9-1 and by (8.23), we deduce that there is no burning tree in $[\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0 + \alpha) \rfloor - \mathbf{k}_{\lambda,\pi}]$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$.

On the one hand, on $\Omega(\alpha, \gamma, \lambda, \pi)$, recall (8.26) and Lemma 4.2, there holds that

$$\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi} - T_l)}^{l,+} < \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor.$$

Thus the right front of the fire l does not reach $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ before $\mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})$. Hence, no fire coming from the left can affect the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ during the considered time interval.

On the other hand, no fire coming from the right can affect $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ before $\mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})$. Indeed, since there is no fire in $[\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor, \lfloor \mathbf{n}_\lambda X_l^+(s_0 + \alpha) \rfloor - \mathbf{k}_{\lambda,\pi}]$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$, we deduce, by $\Omega(\alpha, \gamma, \lambda, \pi)$, that if a fire affect the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ during the time interval $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})]$, it is necessarily a left front. But, by construction, if $X_{l'}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^-$, for some $l' \leq q$, then $X_l^+(s_0) \leq X_{l'}^-(s_0)$ (because $s_0 \leq T_l^{D,+}$). By (8.26) and Lemma 4.2, we then have

$$\lfloor \mathbf{n}_\lambda X_{l'} \rfloor + i_{\mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi} - T_{l'})}^{l',-} > \lfloor \mathbf{n}_\lambda X_{l'}^-(s_0) \rfloor \geq \lfloor \mathbf{n}_\lambda X_l^-(s_0) \rfloor.$$

Hence, no fire coming from the right can affect $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ during the considered time interval. \square

The two following lemmas are the keys of this Stage. The first of them insure that a fire indeed propagates. The second insure that a fire is stopped when it meet a microscopic zone.

Lemma 8.12. *Let $s_0 \in [T_q^k + \alpha, T_q^{k+1} + \alpha]$. We work on $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}$.*

1. *In the limit process, if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+$ for some $l \leq q$ in such a way that $s_0 \leq T_l^{D,+}$ and $F_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0 + \alpha))$, then*

$$\eta_{\mathbf{a}_\lambda T_l + T_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda,\pi}(i) = 2$$

for all $i \in [\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+}, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]$.

2. *In the limit process, if $X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^-$ for all $y \in (X_l^-(s_0 + \alpha), X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}))$, then $\eta_{\mathbf{a}_\lambda T_l + T_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda,\pi}(i) = 2$ for all $i \in [\lfloor \mathbf{n}_\lambda X_l^-(s_0) \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}, \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,-}]$.*

We have the propagation of the fire l only to $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}$. Unfortunately, in the case where $s_0 = T_q^{k+1} = T_l^{D,+}$ and $X_l^+(T_q^{k+1}) = X_q^{k+1} = X_l^{D,+}$ (that is if the right front of the fire l is stopped at time T_q^{k+1} in the limit process), we can not say anything more on the discrete process, due to (8.24). We will show below (see Lemma 8.13) that, in this special case, the zone $[\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda]$ is actually completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})$. This will imply that the fire propagates indeed until $\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})$, thanks to (8.24).

Proof. Lemma 8.11 shows that the site $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$ is not affected by a fire during $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(s_0 - \mathbf{e}_{\lambda,\pi})]$. Hence, no fire coming from the right affect the zone $[\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor]$ during the time interval $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(s_0 - \mathbf{v}_{\lambda,\pi})]$ and, conversely, the right front of the fire l does not affect the zone on the right of $\lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor$. Since $\eta_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+}) = 2$, thanks to Lemma 8.9-2, it then suffices to show that for all $i \in [\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}]$,

$$\eta_{\mathbf{a}_\lambda T_l + T_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda,\pi}(i) = 1$$

i.e. the site i is occupied just before that the right front of the fire l reaches i .

Observe that by construction, in the limit process, no fire affect the site $i/\mathbf{n}_\lambda \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0))$ during $(T_q^k + 4\mathbf{v}_{\lambda,\pi}, T_l(i/\mathbf{n}_\lambda))$ whence in the discrete process, no fire can affect the site $i \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} \rrbracket$ during $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda T_l + T_{i-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l]$. All this implies that for all $i/\mathbf{n}_\lambda \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0))$, we have

$$\tau_{T_l(i/\mathbf{n}_\lambda) - (i/\mathbf{n}_\lambda)} = \tau_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(i/\mathbf{n}_\lambda)$$

while for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} \rrbracket$ we have

$$\rho_{T_l + T_{i-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l / \mathbf{a}_\lambda - (i)}^{\lambda,\pi} = \rho_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}(i).$$

Step 1. Here we show that for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rfloor + \mathbf{k}_{\lambda,\pi} \rrbracket$, we have $\eta_{\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l}^{\lambda,\pi}(j) = 1$.

In Lemma 8.9-2 we have proved that $\eta_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(j) = 1$ for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, \lfloor \mathbf{n}_\lambda X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rfloor + \mathbf{k}_{\lambda,\pi} \rrbracket$. The result follows from the previous observation.

Step 2. Here we show that for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rfloor + \mathbf{k}_{\lambda,\pi} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} \rrbracket \setminus \cup_{y \in \mathcal{B}_M^D}[y]_{\lambda,\pi}$, we have $\eta_{\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l}^{\lambda,\pi}(j) = 1$.

Indeed, on the one hand, $Z_{T_l(j/\mathbf{n}_\lambda) - (j/\mathbf{n}_\lambda)} = 1$, then $T_l(j/\mathbf{n}_\lambda) - \tau_{T_l(j/\mathbf{n}_\lambda) - (j/\mathbf{n}_\lambda)} > 1$ whence

$$\tau_{T_l(j/\mathbf{n}_\lambda) - (j/\mathbf{n}_\lambda)} < T_l(j/\mathbf{n}_\lambda) - 1 - 3\alpha,$$

thanks to $\Omega_M(\alpha)$. On the other hand, recalling that there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \rfloor + \mathbf{k}_{\lambda,\pi} + 1, \lfloor \mathbf{n}_\lambda X_l^+(s_0) \rfloor \rrbracket$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$ (thus $j \notin \cup_{x \in \mathcal{X}_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}} \langle x \rangle_{\lambda,\pi}$) and since $j \notin \cup_{x \in \mathcal{B}_M^D}[x]_{\lambda,\pi}$, we deduce from Lemma 8.5 and by (8.29) that

$$\rho_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}(j) \leq \tau_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(j/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda,\pi}.$$

All this implies that

$$\rho_{T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l / \mathbf{a}_\lambda - (j)}^{\lambda,\pi} \leq T_l(j/\mathbf{n}_\lambda) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi}.$$

Recalling that $T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l / \mathbf{a}_\lambda \geq T_l(j/\mathbf{n}_\lambda) - \mathbf{e}_{\lambda,\pi}$, thanks to (8.29), and $\mathbf{e}_{\lambda,\pi} < \alpha$, we conclude using $\Omega_3^S(\lambda, \pi)$ that the site j is occupied at time $\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l$.

Step 3. Here we show that for all $y \in \mathcal{B}_M^D \cap (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0))$, for all $j \in [y]_{\lambda,\pi}$, there holds $\eta_{\mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(j) = 1$. This will conclude Lemma 8.12 since $\mathbf{a}_\lambda T_l + T_{j-\lfloor \mathbf{n}_\lambda X_l \rfloor}^l \geq \mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi})$ for all $j \in [y]_{\lambda,\pi}$, thanks to (8.27).

Preliminary considerations. Let $y \in \mathcal{B}_M^D \cap (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_l^+(s_0))$. Since $X_l^+(s_0) \leq X_l^{D,+}$, we have $y \leq X_l^{D,+} - 3\alpha/p$. We may assume $X_l^+(s_0) \geq y + \alpha/p$, by $\Omega_M(\alpha)$. We know that $\tilde{H}_{T_l(y)-(y)} = 0$, whence $H_{T_l(y)-(y)} = 0$ and $Z_{T_l(y)-(y)} = Z_{T_l(y)-(y_+)} = Z_{T_l(y)-(y_-)} = 1$. This implies that $T_l(y) \geq 1$ (because $Z_t(y) = t$ for all $t < 1$ and all $y \in [-A, A]$).

As pointed out in Step 2, we have, setting $j_g = \lfloor \mathbf{n}_\lambda y \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} - 1$ and observing that $T_l + T_{j_g - \lfloor \mathbf{n}_\lambda X_l \rfloor}^l / \mathbf{a}_\lambda \geq T_l(y) - 4\mathbf{v}_{\lambda,\pi} \geq T_q^k + 4\mathbf{v}_{\lambda,\pi}$,

$$\rho_{T_l(y) - 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}(j_g) \leq T_l(j_g/\mathbf{n}_\lambda) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi} = T_l(y) - 1 - 3\alpha + \mathbf{e}_{\lambda,\pi} - p \frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} + 1}{\mathbf{n}_\lambda}.$$

Using a similar argument for $j_d = \lfloor \mathbf{n}_\lambda y \rfloor + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi} + 1$, we conclude that no match falling outside $[y]_{\lambda,\pi} = \llbracket j_g + 1, j_d - 1 \rrbracket$ can affect $[y]_{\lambda,\pi}$ during $(\mathbf{a}_\lambda(T_l(y) - 1 - \alpha), \mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda,\pi}))$, because

$$\rho_{T_l(y) - 4\mathbf{v}_{\lambda,\pi}}^{\lambda,\pi}(j_g) + 2\varepsilon_\lambda + 2 \frac{\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}}{\mathbf{a}_\lambda \pi} \leq T_l(y) - 1 - \alpha$$

and because to affect a site $i \in [y]_{\lambda, \pi}$, a match falling outside $[y]_{\lambda, \pi}$ needs to cross j_d or j_g and thus must verify, recall Lemma 8.5,

$$\rho_{T_l(y)-4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(i) \leq (\rho_{T_l(y)-4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(j_g/\mathbf{n}_\lambda) \vee \rho_{T_l(y)-4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}(j_d/\mathbf{n}_\lambda)) + 2(\kappa_{\lambda, \pi}^0 + \mathbf{e}_{\lambda, \pi}).$$

Case 1. First assume that $y \in \mathcal{B}_M^2$. Then we know that no match has fallen on $[y]_{\lambda, \pi}$ during $[0, \mathbf{a}_\lambda T_l(y))$. Due to the preliminary considerations, we deduce that no fire at all has concerned $[y]_{\lambda, \pi}$ during $(\mathbf{a}_\lambda(T_l(y) - 1 - \alpha), \mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda, \pi}))$. Using $\Omega_3^S(\lambda, \pi)$, we conclude that $[y]_{\lambda, \pi}$ is completely occupied at time $\mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda, \pi})$.

Case 2. Assume that $y = X_m \in \mathcal{B}_M$ with $m \geq q + 1$. Then we know that no match has fallen on $[X_m]_{\lambda, \pi}$ during $[0, \mathbf{a}_\lambda T_l(X_m)) \subset [0, \mathbf{a}_\lambda T_m)$. We conclude as in Case 1 using $\Omega_3^S(\lambda, \pi)$ that the zone $[X_m]_{\lambda, \pi}$ is completely occupied at time $\mathbf{a}_\lambda(T_l(y) - 4\mathbf{v}_{\lambda, \pi})$.

Case 3. Assume that $y = X_m \in \mathcal{B}_M$ with $m \leq q$ and $Z_{T_m-}(X_m) = 1$, so that there already has been a macroscopic fire in $[X_m]_{\lambda, \pi}$ (at time $\mathbf{a}_\lambda T_m$). There is no more burning tree in $[X_m]_{\lambda, \pi}$ at time $\mathbf{a}_\lambda(T_m + 4\mathbf{v}_{\lambda, \pi})$, thanks to $\Omega_{\lambda, \pi}^{P, T}(X_m, T_m)$ and (8.28). Since $Z_{T_m}(X_m) = 0$ and $Z_{T_l(X_m)-}(X_m) = 1$, we deduce that $T_l(X_m) - T_m \geq 1$, whence $T_l(X_m) - T_m \geq 1 + 3\alpha$ as usual. We conclude as in case 1 that no fire at all has concerned $[X_m]_{\lambda, \pi}$ during $(\mathbf{a}_\lambda(T_l(X_m) - 1 - \alpha), \mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda, \pi}))$, which implies the claim by $\Omega_3^S(\lambda, \pi)$.

Case 4. Assume that $y = X_m \in \mathcal{B}_M$ with $m \leq q$ and $Z_{T_m-}(X_m) < 1$ and $T_l(X_m) - T_m \geq 1$, whence $T_l(X_m) - T_m \geq 1 + 3\alpha$ due to $\Omega_M(\alpha)$. Then there already has been a microscopic fire in $[X_m]_{\lambda, \pi}$ (at time $\mathbf{a}_\lambda T_m$). There is no more burning tree in $[X_m]_{\lambda, \pi}$ at time $\mathbf{a}_\lambda(T_m + 4\mathbf{v}_{\lambda, \pi})$, thanks to $\Omega_{\lambda, \pi}^{P, T}(X_m, T_m)$ and (8.28). No match falls on $[X_m]_{\lambda, \pi}$ during $(\mathbf{a}_\lambda(T_m + 4\mathbf{v}_{\lambda, \pi}), \mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda, \pi})) \supset (\mathbf{a}_\lambda(T_l(X_m) - 1 - \alpha), \mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda, \pi}))$ and we conclude as in case 1.

Case 5. Assume that $y = X_m \in \mathcal{B}_M$ with $m \leq q$ and $Z_{T_m-}(X_m) < 1$ and $T_l(X_m) - T_m < 1$, whence $T_l(X_m) - T_m \leq 1 - 3\alpha$ due to $\Omega_M(\alpha)$. There has been a microscopic fire in $[X_m]_{\lambda, \pi}$ (at time $\mathbf{a}_\lambda T_m$). Since $H_{T_l(X_m)}(X_m) = 0$, we deduce that $T_m + Z_{T_m-}(X_m) \leq T_l(X_m)$, whence $T_m + Z_{T_m-}(X_m) \leq T_l(X_m) - 3\alpha$ by $\Omega_M(\alpha)$. We define $\mathcal{M} = (i_0; t_0, T_m)$ as in Lemma 8.10.

Consider the zone $C^P := C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_m, T_m)) \subset (X_m)_\lambda$ destroyed by the match falling on $[\mathbf{n}_\lambda X_m]$ at time $\mathbf{a}_\lambda T_m$. This zone is completely occupied at time $\mathbf{a}_\lambda(T_m + \Theta_{\mathcal{M}}^{\lambda, \pi, m})$: this follows from the definition of $\Theta_{\mathcal{M}}^{\lambda, \pi, m}$ (see Lemma 8.2), from Lemma 8.10 and from the preliminary considerations (because $T_m \geq T_l(X_m) - 1 - \alpha$). Using $\Omega_4^S(\gamma, \lambda, \pi)$, we deduce that $T_m + \Theta_{\mathcal{M}}^{\lambda, \pi, m} \leq T_m + Z_{T_m-}(X_m) + \gamma < T_l(X_m) - 2\alpha$, since $\gamma < \alpha$. Hence C^P is completely occupied at time $\mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda, \pi})$.

Consider now $i \in [X_m]_{\lambda, \pi} \setminus C^P$. Then i has not been killed by the fire starting at $[\mathbf{n}_\lambda X_m]$. Thus i cannot have been killed during $(\mathbf{a}_\lambda(T_l(X_m) - 1 - \alpha), \mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda, \pi}))$ (due to the preliminary considerations) and we conclude, using $\Omega_3^S(\lambda, \pi)$, that i is occupied at time $\mathbf{a}_\lambda(T_l(X_m) - 4\mathbf{v}_{\lambda, \pi})$. This implies the claim. \square

We now examine the process at time $\mathbf{a}_\lambda T_q^{k+1}$ around $[\mathbf{n}_\lambda X_q^{k+1}]$ in the case where the fire is stopped by a microscopic zone (in the limit process).

Lemma 8.13. *On $\Omega(\alpha, \gamma, \lambda, \pi) \cap \Omega_{T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$, if $F_{T_q^{k+1}}(X_q^{k+1}) \leq 1$, there exists $i \in (X_q^{k+1})_\lambda$ such that*

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(i) = 0 \text{ for all } s \in [T_q^{k+1} - \mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \mathbf{v}_{\lambda, \pi}].$$

Furthermore,

- (i) if $X_q^{k+1} = X_l^+(T_q^{k+1})$ for some $l \leq q$, then the zone $[[\mathbf{n}_\lambda X_q^{k+1}] - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, [\mathbf{n}_\lambda X_q^{k+1}] - \mathbf{m}_\lambda]$ is completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})$;
- (ii) if $X_q^{k+1} = X_l^-(T_q^{k+1})$ for some $l \leq q$, then the zone $[[\mathbf{n}_\lambda X_q^{k+1}] + \mathbf{m}_\lambda, [\mathbf{n}_\lambda X_q^{k+1}] + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda, \pi}]$ is completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})$.

Proof. We have $\tilde{H}_{T_q^{k+1}}(X_q^{k+1}) > 0$: in the limit process, a fire is stopped in X_q^{k+1} at time T_q^{k+1} by a microscopic zone. Without loss of generality, we assume that $Z_{T_q^{k+1}-}(X_q^{k+1}-) = 1$. We have either $H_{T_q^{k+1}-}(X_q^{k+1}) > 0$ or $Z_{T_q^{k+1}-}(X_q^{k+1}+) < 1$. Clearly, $X_q^{k+1} = X_m \in \mathcal{B}_M$ for some $m \leq q$, with $Z_{T_m-}(X_m) < 1$ (else, we would have $H_{T_q^{k+1}}(X_q^{k+1}) = 0$ and $Z_{T_q^{k+1}-}(X_q^{k+1}-) = Z_{T_q^{k+1}-}(X_q^{k+1}+)$). We define $\mathcal{M} = (i_0; t_0, T_m)$ as in Lemma 8.10.

By construction, there is $l \in \{1, \dots, q\}$ such that $X_m = X_l^+(T_q^{k+1})$. Hence, $T_q^{k+1} = T_l^{D,+}$ and $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+$ with $F_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_q^{k+1} + \alpha/p)$. By Lemma 8.9, we deduce that there is no burning tree in $\llbracket [\mathbf{n}_\lambda X_l] + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_l)}^{l,+} + 1, [\mathbf{n}_\lambda X_q^{k+1}] \rrbracket$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$ whence by Lemma 8.11, that the site $[\mathbf{n}_\lambda X_q^{k+1}]$ is not affected by a fire during $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$. The site $[\mathbf{n}_\lambda X_q^{k+1}] - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} - 1$ is not been affected by any fire during the time interval $(\mathbf{a}_\lambda(T_q^{k+1} - 1 - 2\alpha), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi}))$, recall Step 2 in the proof of Lemma 8.12.

Case 1. Assume first that $H_{T_q^{k+1}-}(X_q^{k+1}) > 0$. Then by construction, there holds $T_m + Z_{T_m-}(X_m) > T_q^{k+1} > T_m$, whence by $\Omega_M(\alpha)$, $T_m + Z_{T_m-}(X_m) > T_q^{k+1} + 2\alpha > T_m + 4\alpha$.

We deduce from Lemma 8.2 that there is a vacant site in

$$C^P = C^P((\zeta_t^{\lambda,\pi,\mathcal{M},m}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, T_m)) = \llbracket i^g, i^d \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$$

during the time interval $[\mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0), \mathbf{a}_\lambda(T_m + \Theta_{\mathcal{M}}^{\lambda,\pi,m})]$ (by definition of $\Theta_{\mathcal{M}}^{\lambda,\pi,m}$). By Lemma 8.10 and with our coupling (recall that seeds fall on $(X_m)_\lambda$ according to the processes $(N_t^{S,m}(i - \lfloor \mathbf{n}_\lambda X_m \rfloor))_{t \geq 0, i \in (X_m)_\lambda}$), we deduce that there is also a vacant site in $\llbracket [\mathbf{n}_\lambda X_m] + i^g, [\mathbf{n}_\lambda X_m] + i^d \rrbracket \subset (X_m)_\lambda$ during $[\mathbf{a}_\lambda(T_m + \kappa_{\lambda,\pi}^0), \mathbf{a}_\lambda(T_m + \Theta_{\mathcal{M}}^{\lambda,\pi,m})]$. But by $\Omega_4^{S,P}(\gamma, \lambda, \pi)$, we see that $\Theta_{\mathcal{M}}^{\lambda,\pi,m} \geq Z_{T_m-}(X_m) - \gamma$ whence $T_m + \Theta_{\mathcal{M}}^{\lambda,\pi,m} \geq T_m + Z_{T_m-}(X_m) - \gamma > T_q^{k+1} + 2\alpha - \gamma > T_q^{k+1} + \mathbf{v}_{\lambda,\pi}$ since $\gamma < \alpha$ and $\mathbf{v}_{\lambda,\pi} < \alpha$. All this implies that there is a vacant site in $C^P \subset (X_m)_\lambda$ during $[\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda,\pi})]$.

Since the match falling on $[\mathbf{n}_\lambda X_m]$ does not affect the zone outside $(X_m)_\lambda$, we deduce from the preliminary considerations that the zone $\llbracket [\mathbf{n}_\lambda X_q^{k+1}] - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi}, [\mathbf{n}_\lambda X_q^{k+1}] - \mathbf{m}_\lambda \rrbracket$ is not affected by any fire during $[\mathbf{a}_\lambda(T_q^{k+1} - 1 - \alpha), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$, which implies the claim by $\Omega_3^S(\lambda, \pi)$.

Case 2. Assume that $H_{T_q^{k+1}-}(X_m) = 0$. Then by construction, there holds $T_q^{k+1} - [T_m - Z_{T_m-}(X_m)] \geq 1$, whence $T_q^{k+1} - [T_m - Z_{T_m-}(X_m)] \geq 1 + 3\alpha$. Since $H_{T_q^{k+1}-}(X_m) = 0$, we have $Z_{T_q^{k+1}-}(X_m+) < 1 = Z_{T_q^{k+1}-}(X_m-)$ and $T_m + Z_{T_m-}(X_m) \leq T_q^{k+1}$, so that $T_m + Z_{T_m-}(X_m) \leq T_q^{k+1} - 3\alpha$.

We aim to use the event $\Omega_1^{S,P}(\lambda, \pi)$. We recall that $t_0 = T_m - Z_{T_m-}(X_m) = \tau_{T_m-}(X_m)$. Observe that $Z_{t_0-}(X_m) = Z_{t_0-}(X_m-) = Z_{t_0-}(X_m+) = 1$ because there is no match falling on x during $[0, T_m)$.

Set now $t_1 = T_m$. Observe that $0 < t_1 - t_0 < 1$ (because $Z_{T_m}(X_m) < 1$). Necessarily, $Z_{t-}(x_+)$ has jumped to 0 at least one time between t_0 and $T_q^{k+1}-$ (else, one would have $Z_{T_q^{k+1}-}(x_+) = 1$, since $T_q^{k+1} - t_0 \geq 1$ by assumption) and this jump occurs after $t_0 + 1 > t_1$ (since a jump of $Z_{t-}(x_+)$ requires that $Z_{t-}(x_+) = 1$, and since for all $t \in (t_0, t_0 + 1)$, $Z_{t-}(x_+) = t - t_0 < 1$).

We thus may denote by $t_2 < t_3 < \dots < t_K$, for some $K \geq 2$, the successive times of jumps of the process $(Z_{t-}(x_-), Z_{t-}(x_+))$ during $(t_0 + 1, T_q^{k+1})$. Then we observe that $Z_{t-}(x_+)$ and $Z_{t-}(x_-)$ do never jump to 0 at the same time during (t_0, T_q^{k+1}) (else it would mean that x is crossed by a fire at some time u , whence necessarily $H_r(x) = 0$ and $Z_{r-}(x+) = Z_{r-}(x-)$ for all $r \in [u, T_q^{k+1}]$).

Furthermore there is always at least one jump of $(Z_{t-}(x_-), Z_{t-}(x_+))$ of any time interval of length 1 (during (t_0, T_q^{k+1})), because else, $Z_{t-}(x_-)$ and $Z_{t-}(x_+)$ would both become to be equal to 1 and thus would remain equal forever.

Finally, observe that two jumps of $Z_{t-}(x_+)$ cannot occur in a time of length 1 (since a jump of $Z_{t-}(x_+)$ requires that $Z_{t-}(x_+) = 1$) and the same thing holds for $Z_{t-}(x_-)$.

Consequently the family $\mathcal{P} = \{t_0, \dots, t_K\}$ necessarily satisfies the condition (PP1) of Subsection 8.3.

For each $l \in \{0, 2, \dots, K\}$, there is a unique (thanks to $\Omega_M(\alpha)$) $k_l \in \llbracket 0, q \rrbracket$ such that $t_l = T_{k_l}(X_m)$. We set, for all $l \in \{0, 2, \dots, K\}$,

$$i_l = \lfloor \mathbf{n}_\lambda X_{k_l} \rfloor + i_{\mathbf{a}_\lambda(t_l - \mathbf{v}_{\lambda, \pi} - T_{k_l})}^{k_l, +} - \lfloor \mathbf{n}_\lambda X_m \rfloor$$

if the jump at time t_l is a jump of $Z_{t-}(X_m-)$ (that is if $x = X_{k_l}^+(t_l)$) and

$$i_l = \lfloor \mathbf{n}_\lambda X_{k_l} \rfloor + i_{\mathbf{a}_\lambda(t_l - \mathbf{v}_{\lambda, \pi} - T_{k_l})}^{k_l, -} - \lfloor \mathbf{n}_\lambda X_m \rfloor$$

if the jump at time t_l is a jump of $Z_{t-}(X_m+)$ (that is if $x = X_{k_l}^-(t_l)$). Set for example $i_0 = 0$ if $t_0 = 0$. We also put $\varepsilon = -1$ if $x = X_{l_2}^+(t_2)$ and $\varepsilon = 1$ else. We thus may denote $\mathcal{I} = (\varepsilon; i_{k_0}, i_{k_2}, \dots, i_{k_K})$. Clearly, \mathcal{I} satisfies (PP2), thanks to (8.24).

All this implies that $\mathfrak{P} = (\mathcal{P}, \mathcal{I})$ satisfies (PP).

Next, there holds that $t_2 - t_1 < Z_{T_m-}(X_m) = t_1 - t_0$, because else, we would have $H_{t_2-}(X_m) = 0$ and thus the fire k_2 would cross X_m , so that $Z_{t-}(x_+)$ and $Z_{t-}(x_-)$ would remain equal forever. Furthermore, we have $0 < T_q^{k+1} - t_K < 1$ because else, we would have $Z_{T_q^{k+1}}(X_m-) = Z_{T_q^{k+1}}(X_m+) = 1$.

Finally, we check that

$$(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i))_{t \in [t_0 - \mathbf{v}_{\lambda, \pi}, t_K + 4\mathbf{v}_{\lambda, \pi}], i \in (X_m)_\lambda} = (\zeta_{\mathbf{a}_\lambda t}^{\lambda, \pi, \mathfrak{P}, m}(i - \lfloor \mathbf{n}_\lambda x \rfloor))_{t \in [t_0 - \mathbf{v}_{\lambda, \pi}, t_K + 4\mathbf{v}_{\lambda, \pi}], i \in (X_m)_\lambda} \quad (8.32)$$

this last process being built with the family of seed processes $(N_t^{S, m}(i))_{t \geq 0, i \in \mathbb{Z}}$ and the family of propagation processes $(N_t^{P, m}(i))_{t \geq 0, i \in \mathbb{Z}}$ as in Subsection 8.3. We do *e.g.* it in the case where $\varepsilon = -1$ and $t_0 > 1$, the other cases being treated similarly.

Observe that for all $l \in \{0, 2, \dots, K\}$ there holds $t_l = T_{k_l}(X_m) = T_{k_l}^{D, +}$ (if $X_m = X_{k_l}^+(t_l)$) or $T_{k_l}^{D, -}$ (if $X_m = X_{k_l}^-(t_l)$). Hence, since $T_q^k + 4\mathbf{v}_{\lambda, \pi} \geq T_l + \mathbf{v}_{\lambda, \pi}$, we have

$$\eta_{\mathbf{a}_\lambda(t_l - \mathbf{v}_{\lambda, \pi})}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_m \rfloor + i_l) = 2 \quad (8.33)$$

for all $l \in \{0, 2, \dots, K\}$, thanks to $\Omega_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$.

We already have checked in Lemma 8.10 that $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda, \pi, \mathfrak{P}, m}(i - \lfloor \mathbf{n}_\lambda x \rfloor))_{t \geq 0, i \in \mathbb{Z}}$ are equal on $(X_m)_\lambda$ during the time interval $[\mathbf{a}_\lambda(t_0 - \mathbf{v}_{\lambda, \pi}), \mathbf{a}_\lambda(T_m + \kappa_{\lambda, \pi}^0)]$. Using similar argument, observing that seeds fall on $[X_m]_{\lambda, \pi}$ and fires spreads through $[X_m]_{\lambda, \pi}$ according to the same processes and using (8.33), we easily deduce that (8.32) holds on $\Omega(\alpha, \gamma, \lambda, \pi)$.

We thus can use $\Omega_1^{S, P}(\lambda, \pi)$ and conclude that

- there is $i \in (X_m)_\lambda$ with $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(i) = 0$ for all $t \in [T_q^{k+1} - \mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \mathbf{v}_{\lambda, \pi}] \subset [t_K + 2\mathbf{v}_{\lambda, \pi}, t_K + 1 - \mathbf{v}_{\lambda, \pi}]$;
- no fire coming from the right can affect the zone on the left of $\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda$ during the time interval $[\mathbf{a}_\lambda T_m, \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})]$ (because the fire are stopped by vacant site in $(X_m)_\lambda$). Hence, to affect the zone $\llbracket \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda \rrbracket$ during this time interval, a fire must come from the left and thus must affect the site $\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} - 1$. We deduce from the preliminary considerations that the zone $\llbracket \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda \rrbracket$ is not affected by any fire during $[\mathbf{a}_\lambda(T_q^{k+1} - 1 - \alpha), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})]$ which implies the claim by $\Omega_3^S(\lambda, \pi)$. \square

We deduce the following corollary, which is the goal of Stage 2.

Corollary 8.14. *On $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$ implies $\Omega_{T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$.*

Proof. We have to prove that for $l \leq q$,

- (a) if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ and if $T_l^{D, +} \neq T_q^{k+1}$, then $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s - T_l)}^{l, +}) = 2$ for all $s \in [T_q^k + 4\mathbf{v}_{\lambda, \pi}, T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}]$;

- (b) if $X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^-$ and if $T_l^{D,-} \neq T_q^{k+1}$, then $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,-}) = 2$ for all $s \in [T_q^k + 4\mathbf{v}_{\lambda,\pi}, T_q^{k+1} + 4\mathbf{v}_{\lambda,\pi}]$;
- (c) if $X_l^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+$ and if $T_l^{D,+} = T_q^{k+1}$, then $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,+}) = 2$ for all $s \in [T_q^k + 4\mathbf{v}_{\lambda,\pi}, T_q^{k+1} - \mathbf{v}_{\lambda,\pi}]$ and there is $s \in [T_q^{k+1} - \mathbf{v}_{\lambda,\pi}, T_q^{k+1} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,+}) = 0$;
- (d) if $X_l^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^-$ and if $T_l^{D,-} = T_q^{k+1}$, then $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,-}) = 2$ for all $s \in [T_q^k + 4\mathbf{v}_{\lambda,\pi}, T_q^{k+1} - \mathbf{v}_{\lambda,\pi}]$ and there is $s \in [T_q^{k+1} - \mathbf{v}_{\lambda,\pi}, T_q^{k+1} + \mathbf{v}_{\lambda,\pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l,-}) = 0$.

All this will imply the result (observe that only these four cases may occur).

Observe that either $F_{T_q^{k+1}}(X_q^{k+1}) = 2$ (*i.e.* two fires meet at time T_q^{k+1}) or $F_{T_q^{k+1}}(X_q^{k+1}) \leq 1$ (*i.e.* a fire is stopped by a microscopic zone).

Step 1. We start by studying the case where $F_{T_q^{k+1}}(X_q^{k+1}) = 2$. There are l_1 and l_2 such that $X_{l_1}^+(T_q^{k+1}) = X_q^{k+1} = X_{l_2}^-(T_q^{k+1})$. In this Step, we prove (c) for the fire l_1 and (d) for the fire l_2 .

By construction, we have $X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^+$ and $X_{l_2}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}^-$ with $F_{T_q^k + 4\mathbf{v}_{\lambda,\pi}}(y) = 0$ for all $y \in (X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}), X_{l_2}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}))$ and $X_{l_2}^-(T_q^k + 4\mathbf{v}_{\lambda,\pi}) - X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda,\pi}) = 2(T_q^{k+1} - T_q^k - 4\mathbf{v}_{\lambda,\pi})/p \geq 5\alpha/p$.

We first prove that $\eta_{\mathbf{a}_\lambda s}^{\lambda,\pi}(\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(s-T_{l_1})}^{l_1,+}) = 2$ for all $s \in [\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})]$. Equivalently, we prove that

$$\eta_{\mathbf{a}_\lambda T_{l_1} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}}^{\lambda,\pi}(j) = 2$$

for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+}, \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+} \rrbracket$.

Firstly, Lemma 8.12 with $s_0 = T_q^{k+1}$ directly implies that $\eta_{\mathbf{a}_\lambda T_{l_1} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}}^{\lambda,\pi}(j) = 2$ for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} \rrbracket$.

Secondly, we prove that

$$\eta_{\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(i) = 1 \text{ for all } i \in [X_q^{k+1}]_{\lambda,\pi}.$$

This will complete the claim, using similar arguments as in Lemma 8.12 since there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+} + 1, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_2})}^{l_2,-} + 1 \rrbracket$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi})$, thanks to Lemma 8.9 and since $\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+} \leq \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda$ and $\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi} - T_{l_2})}^{l_2,-} \geq \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor + \mathbf{m}_\lambda$, thanks to $\Omega^{P,T}(\lambda, \pi)$ and (8.24).

No fire can affect the zone $[X_q^{k+1}]_{\lambda,\pi}$ during $[\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi}), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})]$, thanks to (8.27) and to Lemma 8.9, (which implies that there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+} + 1, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda,\pi} - T_{l_2})}^{l_2,-} - 1 \rrbracket$). By construction, we have $Z_{T_q^{k+1}-}(X_q^{k+1}) = Z_{T_q^{k+1}-}(X_q^{k+1}+) = Z_{T_q^{k+1}-}(X_q^{k+1}-) = 1$, whence $T_q^{k+1} - \tau_{T_q^{k+1}}(X_q^{k+1}) \geq 1$ and $T_q^{k+1} - \tau_{T_q^{k+1}}(X_q^{k+1}) \geq 1 + 3\alpha$ by $\Omega_M(\alpha)$. Since no match has fallen on $X_q^{k+1} \in \mathcal{B}_M$ during $[0, T_q^{k+1}]$, using similar argument as in Case 1 Step 3 in the proof of Lemma 8.12, we then deduce that for all $j \in [X_q^{k+1}]_{\lambda,\pi}$,

$$\rho_{\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(j) \leq T_q^{k+1} - 1 - \alpha,$$

which implies the claim by $\Omega_3^S(\lambda, \pi)$. Same thing of course holds for l_2 .

Furthermore, we have shown that at time $\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})$, the sites $\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi} - T_{l_1})}^{l_1,+}$ and $\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi} - T_{l_2})}^{l_2,-}$ are burning and

$$\eta_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda,\pi})}^{\lambda,\pi}(i) = 1 \tag{8.34}$$

for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi} - T_{l_1})}^{l_1, +} + 1, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi} - T_{l_2})}^{l_2, -} - 1 \rrbracket$.

We next show that the fires are stopped during $[\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi}), \mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda, \pi})]$. Observe that, on $\Omega^{P, T}(\lambda, \pi)$, thanks to (8.25), there is $i_0 \in [X_q^{k+1}]_{\lambda, \pi}$ such that

$$i_0 = \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{T_{i_0+1} - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}^{l_1, +} = \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{T_{i_0-1} - \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor}^{l_2, -}.$$

We deduce from (8.34), that

$$\eta_{\mathbf{a}_\lambda T_{l_1} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}}^{\lambda, \pi}(j) = 2 \text{ for all } j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi} - T_{l_1})}^{l_1, +}, i_0 \rrbracket$$

and

$$\eta_{\mathbf{a}_\lambda T_{l_2} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor}}^{\lambda, \pi}(j) = 2 \text{ for all } j \in \llbracket i_0, \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi} - T_{l_2})}^{l_2, -} \rrbracket.$$

We know that the fire in i_0 propagates at time

$$\mathbf{a}_\lambda T_{l_1} + T_{i_0+1 - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}^{l_1} = \mathbf{a}_\lambda T_{l_2} + T_{i_0-1 - \lfloor \mathbf{n}_\lambda X_{l_2} \rfloor}^{l_2}.$$

Thus, with our coupling and on $\Omega^{P, T}(\lambda, \pi)$, at time $\mathbf{a}_\lambda T_{l_1} + T_{i_0+1 - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}^{l_1}$, either the site $i_0 + 1$ is vacant (because it has been burnt by the fire l_2) or the site $i_0 + 1$ is occupied but has vacant neighbors until it propagates, that is until $\mathbf{a}_\lambda T_{l_1} + T_{i_0+2 - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}^{l_1}$ (because it is a spark for the fire l_2). In any case, since

$$\mathbf{a}_\lambda T_{l_1} + T_{i_0+2 - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}^{l_1} \in [\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi}), \mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda, \pi})],$$

recall (8.29), there is $s_1 \in [T_q^{k+1} - \mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \mathbf{v}_{\lambda, \pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_1})}^{l_1, +}) = 0$. Similarly, we can find $s_2 \in [T_q^{k+1} - \mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \mathbf{v}_{\lambda, \pi}]$ such that $\eta_{\mathbf{a}_\lambda s_2}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_2} \rfloor + i_{\mathbf{a}_\lambda(s_2 - T_{l_2})}^{l_2, +}) = 0$, which completes this Step.

Step 2. Here, we study the case where $F_{T_q^{k+1}}(X_q^{k+1}) \leq 1$ and $X_q^{k+1} \notin \{-A, A\}$. Assume for example that $X_q^{k+1} = X_{l_0}^+(T_q^{k+1})$ for some $l_0 \leq q$. In this Step, we prove (c) for the fire l_0 .

By construction, $X_{l_0}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ and $F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0$ for all $y \in (X_{l_0}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}), X_q^{k+1} + \alpha/p)$.

We first prove that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_0})}^{l_0, +}) = 2$ for all $s \in [\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi}), \mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi})]$. Equivalently, we prove that

$$\eta_{\mathbf{a}_\lambda T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(j) = 2$$

for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} \rrbracket$.

Firstly, using Lemma 8.12 with $s_0 = T_q^{k+1}$, we deduce that $\eta_{\mathbf{a}_\lambda T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(j) = 2$ for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} \rrbracket$.

Secondly, Lemma 8.13-1 shows that the zone $\llbracket \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor - \mathbf{m}_\lambda \rrbracket$ is completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})$. Since no fire coming from the right can affect the zone on the left of $\lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor$ until $\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi})$, we deduce the claim using similar argument as in Lemma 8.12.

Finally, Lemma 8.13 directly imply that there is $j \in (X_q^{k+1})_\lambda$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(j) = 0$ for all $s \in [T_q^{k+1} - \mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \mathbf{v}_{\lambda, \pi}]$. Since

$$\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} + \mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} \geq \lfloor \mathbf{n}_\lambda X_q^{k+1} \rfloor + \mathbf{m}_\lambda,$$

recall (8.25), there is $s \in [T_q^{k+1} - \mathbf{v}_{\lambda, \pi}, T_q^{k+1} + \mathbf{v}_{\lambda, \pi}]$ such that $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_0})}^{l_0, +}) = 0$, as desired.

Step 3. Here we study the case where $X_q^{k+1} \in \{-A, A\}$. Assume for example that $X_q^{k+1} = X_{l_0}^+(T_q^{k+1}) = A$ for some $l_0 \leq q$. In this Step, we prove (c) for the fire l_0 .

This case is very simple: by construction, $X_{l_0}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ and $F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0$ for all $y \in (X_{l_0}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}), A)$.

Since there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} + 1, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})$ (thanks to Lemma 8.9), we deduce, using similar argument as in the proof of Lemma 8.12, that $\eta_{\mathbf{a}_\lambda T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(j) = 2$ for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda A \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} \rrbracket$. The zone $\llbracket \lfloor \mathbf{n}_\lambda A \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ is not affected by any fire during $[\mathbf{a}_\lambda(T_q^{k+1} - 1 - \alpha), \mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})]$ (recall Step 3 in the proof of Lemma 8.12) and no match falls in this zone during $[0, \mathbf{a}_\lambda T]$. We deduce as usual, using $\Omega_3^S(\lambda, \pi)$, that this zone is completely occupied at time $\mathbf{a}_\lambda(T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi})$. Thus, we have

$$\eta_{\mathbf{a}_\lambda T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(j) = 2$$

for all $j \in \llbracket \lfloor \mathbf{n}_\lambda A \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$, which implies the claim since $\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} - \mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} \leq \lfloor \mathbf{n}_\lambda A \rfloor - \mathbf{m}_\lambda$.

We immediately deduce the claim since $\eta_s^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda A \rfloor + 1) = 0$ for all $s \in [0, \infty)$.

Step 4. Here we study the case where $x_0 := X_{l_0}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ with $T_{l_0}^{D, +} \neq T_q^{k+1}$, for some $l_0 \leq q$. We prove (a) for the fire l_0 . By $\Omega_M(\alpha)$, there holds $T_{l_0}^{D, +} \geq T_q^{k+1} + 3\alpha$.

By $\Omega_M(\alpha)$, we have $T_{l_0}^{D, +} \geq T_q^{k+1} + 3\alpha$. If $F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0$ for all $y > x_0$, necessarily $F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0$ for all $y \in (x_0, X_{l_0}^+(T_q^{k+1} + 3\alpha))$. Lemma 8.12 with $s_0 = T_q^{k+1} + 2\alpha$ directly implies the result, since on $\Omega^{P, T}(\lambda, \pi)$, recall (8.23), there holds

$$\begin{aligned} \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} &\leq \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}) \rfloor + \mathbf{k}_{\lambda, \pi} \\ &\leq \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + 2\alpha) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}. \end{aligned}$$

Else, we define

$$x_1 := \inf \left\{ y > x_0 : F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 1 \right\}$$

and distinguish several cases.

Case 1. Assume that $x_1 - x_0 > (T_q^{k+1} - T_q^k + 2\alpha)/p$. Using Lemma 8.12 with $s_0 = T_q^{k+1} + \alpha$, we immediately deduce that

$$\eta_{\mathbf{a}_\lambda T_{l_0} + T_{i - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(i) = 2$$

for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + \alpha) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} \rrbracket$ whence

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_0})}^{l_0, +}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda, \pi}, T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}]$$

because on $\Omega^{P, T}(\lambda, \pi)$, there holds $\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} \leq \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + \alpha) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi}$.

Case 2. Assume that $x_1 - x_0 \leq (T_q^{k+1} - T_q^k + 2\alpha)/p$ but $F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0$ for all $y \in (x_1, x_1 + (T_q^{k+1} - T_q^k + 2\alpha)/p)$. Necessarily $x_1 = X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ for some $l_1 \leq q$.

Using Lemma 8.12 with $s_0 = T_q^{k+1} \leq T_{l_1}^{D, +}$, we deduce that $\eta_{\mathbf{a}_\lambda T_{l_1} + T_{i - \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor}}^{\lambda, \pi}(i) = 2$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_1})}^{l_1, +}, \lfloor \mathbf{n}_\lambda X_{l_1}^+(T_q^{k+1}) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} \rrbracket$. Thus, using (8.27), we deduce

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_1} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_1})}^{l_1, +}) = 2 \text{ for all } s \in [T_{l_1}, T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi}].$$

We now prove that for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} \rrbracket$, we have

$$\eta_{\mathbf{a}_\lambda T_{l_0} + T_{i - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(i) = 2.$$

This will conclude this case.

Firstly, by construction, we have $x_1 > x_0 + 1/p$ whence by $\Omega_M(\alpha)$, $x_1 > x_0 + (1 + 3\alpha)/p$. Thus, using again Lemma 8.12 with $s_0 = T_{l_0}(x_1) - \alpha$, we deduce that

$$\eta_{\mathbf{a}_\lambda T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}}^{\lambda, \pi}(j) = 2$$

for all $j \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda(x_1 - \alpha/p) \rfloor - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda, \pi} \rrbracket$ (recall that $X_{l_0}^+(T_{l_0}(x_1)) = x_1$).

Secondly, observe that $T_{l_1} < T_q^k$ (because else $T_{l_1} = T_q^k$ and $X_{l_1}^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^-$ with $x_0 < X_{l_1}^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) < X_{l_1}^+(T_q^k + 4\mathbf{v}_{\lambda, \pi})$) whence by $\Omega_M(\alpha)$, $T_{l_1} < T_q^k - 3\alpha$. This especially imply that $T_{l_0}(y) \geq T_{l_1}(y) + 1 + 3\alpha$ for all $y \in [x_1 - 3\alpha/p, X_{l_0}^+(T_q^{k+1} + \alpha)]$. Recall that no match falls on any site $y \in (x_1 - 3\alpha/p, X_{l_0}^+(T_q^{k+1} + \alpha))$ during the time interval $(T_q^k - 3\alpha, T_q^{k+1} + \alpha)$. Thus, in the limit process, for all $y \in (x_1 - 3\alpha/p, X_{l_0}^+(T_q^{k+1} + \alpha))$, we have $\tau_{T_{l_0}(y)-}(y) = T_{l_1}(y)$.

Let now $i \in \llbracket \lfloor \mathbf{n}_\lambda(x_1 - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + \alpha) \rfloor \rrbracket$. Observe that there is no burning tree in $\llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} + 1, \lfloor \mathbf{n}_\lambda x_1 \rfloor - \mathbf{k}_{\lambda, \pi} \rrbracket$ at time $\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})$, thanks to Lemma 8.9. Since no match falls on i during $[\mathbf{a}_\lambda(T_{l_1}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}), \mathbf{a}_\lambda(T_q^{k+1} + \alpha)]$, we deduce that no fire at all can affect the site i during the time interval $[\mathbf{a}_\lambda(T_{l_1}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}), \mathbf{a}_\lambda T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}^{l_0}]$ whence

$$\rho_{T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}^{l_0} / \mathbf{a}_\lambda -}^{\lambda, \pi}(i) \leq T_{l_1}(i/\mathbf{n}_\lambda) + \mathbf{e}_{\lambda, \pi}.$$

Thus, for all $i \in \llbracket \lfloor \mathbf{n}_\lambda(x_1 - 2\alpha/p) \rfloor, \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + \alpha) \rfloor \rrbracket$, we have

$$\rho_{T_{l_0} + T_{j - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}^{l_0} / \mathbf{a}_\lambda -}^{\lambda, \pi}(i) \leq T_{l_0}(i/\mathbf{n}_\lambda) - 1 - 3\alpha + \mathbf{e}_{\lambda, \pi}$$

and conclude using $\Omega_3^S(\lambda, \pi)$ that $\eta_{\mathbf{a}_\lambda T_{l_0} + T_{i - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}^{l_0} -}^{\lambda, \pi}(i) = 1$ whence

$$\eta_{\mathbf{a}_\lambda T_{l_0} + T_{i - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}^{l_0}}^{\lambda, \pi}(i) = 2$$

because $\eta_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi})}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_1})}^{l_0, +}) = 2$.

All this implies that $\eta_{\mathbf{a}_\lambda T_{l_0} + T_{i - \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor}^{l_0}}^{\lambda, \pi}(i) = 2$ for all $i \in \llbracket \lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^k + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +}, \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + \alpha) \rfloor \rrbracket$ whence

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_0})}^{l_0, +}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda, \pi}, T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}]$$

since $\lfloor \mathbf{n}_\lambda X_{l_0} \rfloor + i_{\mathbf{a}_\lambda(T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi} - T_{l_0})}^{l_0, +} \leq \lfloor \mathbf{n}_\lambda X_{l_0}^+(T_q^{k+1} + \alpha) \rfloor$. This completes this case.

Case 3. In the general case, by construction, there are $x_0 < x_1 < x_2 < \dots < x_m$ such that, for all $j \in \{0, \dots, m-1\}$,

$$x_j - x_{j+1} \leq (T_q^{k+1} - T_q^k + 2\alpha)/p$$

and

$$F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0 \text{ for all } y \in (x_j, x_{j+1})$$

and finally

$$F_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}(y) = 0 \text{ for all } y \in (x_m, x_m + (T_q^{k+1} - T_q^k - 2\alpha)/p).$$

Clearly, for all $j \in \{1, \dots, m\}$, we have $x_j \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^+$ whence there exists $l_j \in \{1, \dots, q\}$ such that $x_j = X_{l_j}^+(T_q^k + \mathbf{v}_{\lambda, \pi})$.

We first prove, exactly as in case 2, that

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_m} \rfloor + i_{\mathbf{a}_\lambda(s - T_{l_m})}^{l_m, +}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda, \pi}, T_q^{k+1} - 4\mathbf{v}_{\lambda, \pi}].$$

Next, exactly as in Case 2, we can prove that

$$\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_{l_{m-1}} \rfloor + i_{\mathbf{a}_\lambda(s-T_{l_{m-1}})}^{l_{m-1}, +}) = 2 \text{ for all } s \in [T_q^k + 4\mathbf{v}_{\lambda, \pi}, T_q^{k+1} + 4\mathbf{v}_{\lambda, \pi}]$$

and so on.

Step 5. Finally, if $x_0 := X_{l_0}^-(T_q^k + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^k + 4\mathbf{v}_{\lambda, \pi}}^-$ with $T_{l_0}^{D, +} \neq T_q^{k+1}$, for some $l_0 \leq q$, we deduce (b) for the fire l_0 using similar argument as in Step 4.

This completes the proof. \square

STAGE 3.

In this Stage, we treat the time interval $[T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}, T_{q+1}]$. On this time interval, no fire is stopped in the limit process. A match falls in X_{q+1} at time T_{q+1} . The proof of the following lemma is very similar to the proof of the previous Stage.

Lemma 8.15. *On $\Omega(\alpha, \lambda, \gamma, \pi)$, $\Omega_{T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$.*

Sketch of the proof. Observe that $\mathcal{T}_M^D \cap (T_q^{N_q}, T_{q+1}) = \emptyset$. Hence, we have to prove that if $x := X_l^+(T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}}^+$ (or $X_l^-(T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}) \in \chi_{T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}}^-$) for some $l \leq q$, then $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l, +}) = 2$ (or $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_l \rfloor + i_{\mathbf{a}_\lambda(s-T_l)}^{l, -}) = 2$) for all $s \in [T_q^{N_q} + 4\mathbf{v}_{\lambda, \pi}, T_{q+1}]$ (because $T_l^{D, +} > T_{q+1} + 3\alpha$).

We can prove similar lemmas as Lemmas 8.11 and 8.12 replacing T_q^k by $T_q^{N_q}$ and T_q^{k+1} by T_{q+1} . Thus, Lemma 8.15 follows exactly as in Step 4 and Step 5 in the proof of Corollary 8.14. \square

The proof of Lemma 8.4 is completed.

8.5 Proof of Theorem 6.1 for $p > 0$

We finally give the proof of the Theorem 6.1 in the case $p > 0$.

Proof. Let us fix $x_0 \in (-A, A)$, $t_0 \in (0, T)$ and $\varepsilon > 0$. We will prove that with our coupling (see Subsection 8.4.1), when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, there holds that

- (a) $\lim_{\lambda, \pi} \mathbb{P} [\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) > \varepsilon] = 0;$
- (b) $\lim_{\lambda, \pi} \mathbb{P} [\delta_T(D^{\lambda, \pi}(x_0), D(x_0)) > \varepsilon] = 0;$
- (c) $\lim_{\lambda, \pi} \mathbb{P} [|Z_t^{\lambda, \pi}(x_0) - Z_t(x_0)| > \varepsilon] = 0;$
- (d) $\lim_{\lambda, \pi} \mathbb{P} \left[\int_0^T |Z_t^{\lambda, \pi}(x_0) - Z_t(x_0)| dt > \varepsilon \right] = 0;$
- (e) $\lim_{\lambda, \pi} \mathbb{P} [|W_{t_0}^{\lambda, \pi}(x_0) - Z_{t_0}(x_0)| > \varepsilon] = 0$, where

$$W_{t_0}^{\lambda, \pi}(x_0) = \left(\frac{\log(|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 1\}} \right) \wedge 1.$$

These points will clearly imply the result.

First, we introduce the event $\Omega_{A, T}^{x_0, t_0}(\alpha, \lambda, \pi)$ on which

- (i) $x_0 \notin \cup_{y \in \mathcal{B}_M^D \cup \chi_{t_0}} (y - 3\alpha/p, y + 3\alpha/p);$
- (ii) for all $s \in \{T_k(x_0) : k = 1, \dots, n\} \cup \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2$ with $s \leq t_0$, there holds that $t_0 - s > 3\alpha;$
- (iii) if $t_0 \neq 1$, for all $s \in \{T_k(x_0) : k = 1, \dots, n\} \cup \mathcal{T}_M \cup \mathcal{S}_M \cup \mathcal{S}_M^1 \cup \mathcal{S}_M^2$ with $s \leq t_0$, there holds that $|t_0 - (s + 1)| > 3\alpha;$

(iv) if $t_0 \geq 1$, for all $i \in I_A^\lambda$, $N_{\mathbf{a}_\lambda t_0}^{S,\lambda,\pi}(i) - N_{\mathbf{a}_\lambda(t_0-1)}^{S,\lambda,\pi}(i) > 0$;

(v) if $t_c = t_0 - \tau_{t_0-}(x_0) < 1$, there are

$$-\lfloor \lambda^{-(t_c+\alpha)} \rfloor < i_1 < -\lfloor \lambda^{-(t_c-\alpha)} \rfloor < 0 < \lfloor \lambda^{-(t_c-\alpha)} \rfloor < i_2 < \lfloor \lambda^{-(t_c+\alpha)} \rfloor$$

such that

- $N_{\mathbf{a}_\lambda t_0}^{S,0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) - N_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) - \mathbf{v}_{\lambda,\pi})}^{S,0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = 0$ and $N_{\mathbf{a}_\lambda t_0}^{S,0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) - N_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) - \mathbf{v}_{\lambda,\pi})}^{S,0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$;
- $N_{\mathbf{a}_\lambda t_0}^{S,0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) - N_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) + \mathbf{v}_{\lambda,\pi})}^{S,0}(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) > 0$ for all $j \in \llbracket -\lfloor \lambda^{-(t_c-\alpha)} \rfloor, \lfloor \lambda^{-(t_c+\alpha)} \rfloor \rrbracket$.

Since $t_0 - \tau_{t_0-}(x_0) = 1$ occurs with positive probability only if $t_0 = 1$ (and $\tau_{t_0}(x_0) = 0$) the probability of the three first points clearly tend to 1 when α tends to 0. Since $(\tau_t(x_0))_{t \geq 0}$ is independent of $(N_t^{S,\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and since $(\tau_t(x_0))_{t \geq 0} \subset \{T_k(x_0) : k = 1, \dots, n\}$, the probability of the two last points tend to 1 as $\alpha \rightarrow 0$ and $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, thanks to Lemma 8.1-4,6,7. All this implies that for all $\delta > 0$, there is $\alpha > 0$ such that $\mathbb{P} \left[\Omega_{A,T}^{x_0,t_0}(\alpha, \lambda, \pi) \right] > 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(p)$.

Let us now fix $\delta > 0$. We consider $\alpha_0 \in (0, \varepsilon)$, $\gamma_0 \in (0, \alpha_0)$, $\lambda_0 \in (0, 1)$ and $\epsilon_0 \in (0, 1)$ such that for all $\lambda \in (0, \lambda_0)$ and all $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) - p| < \epsilon_0$, we have

$$\mathbb{P} \left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0,t_0}(\alpha_0, \lambda, \pi) \right] > 1 - \delta.$$

We then consider $\lambda_1 \in (0, \lambda_0)$ and $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\lambda \in (0, \lambda_1)$ and all $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) - p| < \epsilon_1$, we have

- $4(\mathbf{v}_{\lambda,\pi} + p(\mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda) \leq \alpha_0$;
- $\alpha_0 + \log(\mathbf{a}_\lambda)/\log(1/\lambda) < \varepsilon$;
- $4(\mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda < \varepsilon$;
- $1/(2\mathbf{m}_\lambda \lambda^{t_c-\varepsilon}) < \delta$ and $1/(2\mathbf{m}_\lambda \lambda^{t_c+\mathbf{v}_{\lambda,\pi}}) < \delta$ if $t_c < 1$.

All this can be done properly by using the fact that $\mathbf{v}_{\lambda,\pi} \rightarrow 0$ and $(\mathbf{m}_\lambda + \mathbf{k}_{\lambda,\pi})/\mathbf{n}_\lambda \rightarrow 0$.

In the rest of the proof, we consider $\lambda \in (0, \lambda_1)$ and $\pi \geq 1$ in such a way that $|\mathbf{n}_\lambda/(\mathbf{a}_\lambda \pi) - p| \leq \epsilon_1$. Observe that, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi)$, there holds that $\tau_{t_0-}(x_0) = \tau_{t_0}(x_0)$ and $[x_0]_{\lambda,\pi} \cap \left(\bigcup_{x \in \mathcal{B}_M^D \cup \mathcal{X}_{t_0}} [x]_{\lambda,\pi} \right) = \emptyset$.

Step 1. We first show that (a) (which holds for an arbitrary value of $t_0 \in (0, T)$) implies (b). Indeed, we have by construction, for any $t \in [0, T]$, $\delta(D_t^{\lambda,\pi}(x_0), D_t(x_0)) < 4A$. Hence, by dominated convergence, (a) implies that $\lim_{\lambda,\pi} \mathbb{E} \left[\delta(D_t^{\lambda,\pi}(x_0), D_t(x_0)) \right] = 0$, whence again by dominated convergence, $\lim_{\lambda,\pi} \mathbb{E} \left[\delta_T(D^{\lambda,\pi}(x_0), D(x_0)) \right] = 0$.

Step 2. Next, (c) implies (d), exactly as in Step 1.

Step 3. Due to Lemma 8.5, we know that, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0,t_0}(\alpha_0, \lambda, \pi)$, since $t_0 > \tau_{t_0}(x_0) + 3\alpha_0$, for all $i \in (x_0)_\lambda$,

$$\left| \rho_{t_0}^{\lambda,\pi}(i) - \tau_{t_0}(x_0) \right| \leq \mathbf{v}_{\lambda,\pi}.$$

For all $i \in (x_0)_\lambda$, since $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) \leq 1$, there holds

$$\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) = \min(N_{\mathbf{a}_\lambda t_0}^{S,\lambda,\pi}(i) - N_{\mathbf{a}_\lambda \rho_{t_0}^{\lambda,\pi}(i)}^{S,\lambda,\pi}(i), 1).$$

Thus, for all $i \in (x_0)_\lambda$,

$$\underline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) \leq \eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) \leq \overline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i)$$

where

$$\begin{aligned}\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) &:= \min(N_{\mathbf{a}_\lambda t_0}^{S, 0}(i) - N_{\mathbf{a}_\lambda(\tau_{t_0}(x_0) + \mathbf{v}_{\lambda, \pi})}^{S, 0}(i), 1), \\ \bar{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) &:= \min(N_{\mathbf{a}_\lambda t_0}^{S, 0}(i) - N_{\mathbf{a}_\lambda(\tau_{t_0}(x_0) - \mathbf{v}_{\lambda, \pi}) \vee 0}^{S, 0}(i), 1).\end{aligned}$$

We also recall that by construction, $(\tau_t(x_0))_{t \geq 0}$ is independent of $(N_t^{S, 0}(i))_{t \geq 0, i \in \mathbb{Z}}$.

Step 4. Here we prove (e). We work on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$. By Step 3 and point (v) of the event $\Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we observe that if $0 < t_c = t_0 - \tau_{t_0}(x_0) < 1$, then

$$\begin{aligned}\llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor - \lfloor \lambda^{-(t_c - \alpha_0)} \rfloor, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \lfloor \lambda^{-(t_c - \alpha_0)} \rfloor \rrbracket &\subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \\ &\subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset C(\bar{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1, \lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2 \rrbracket \\ &\subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor - \lfloor \lambda^{-(t_c + \alpha_0)} \rfloor, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \lfloor \lambda^{-(t_c + \alpha_0)} \rfloor \rrbracket.\end{aligned}$$

Thus, this implies that,

$$|W_{t_0}^{\lambda, \pi}(x_0) - (t_0 - \tau_{t_0}(x_0))| \leq \alpha_0 + \frac{\log(2)}{\log(1/\lambda)} < \varepsilon.$$

If now $t_0 - \tau_{t_0}(x_0) > 1$, then $t_0 - \tau_{t_0}(x_0) > 1 + 3\alpha_0$ thanks to $\Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$. Then Step 3 and point (iv) of $\Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$ imply that $(x_0)_\lambda \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)$ whence $|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 2\mathbf{m}_\lambda$. Consequently,

$$W_{t_0}^{\lambda, \pi}(x_0) \geq 1 - \frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)} > 1 - \varepsilon.$$

It only remains to study what happens when $t_0 = 1$. By construction, we have $\tau_{t_0}(x_0) = 0$ and by Lemma 8.5, we have $\rho_{t_0}^{\lambda, \pi}(i) = 0$ for all $i \in (x_0)_\lambda$. By Step 3 and point (iv) of the event $\Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we deduce as above that $(x_0)_\lambda \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)$ and conclude $|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 2\mathbf{m}_\lambda$ whence

$$W_{t_0}^{\lambda, \pi}(x_0) \geq 1 - \frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)} \geq 1 - \varepsilon.$$

Recalling that $Z_{t_0}(x_0) = (t_0 - \tau_{t_0}(x_0)) \wedge 1$, we have proved that

$$\mathbb{P} \left[|W_{t_0}^{\lambda, \pi}(x_0) - Z_{t_0}(x_0)| < \varepsilon \right] \geq \mathbb{P} \left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi) \right] \geq 1 - \delta,$$

as desired.

Step 5. Here we prove (c). Recall that $Z_{t_0}^{\lambda, \pi}(x_0) = \left(-\frac{\log(1 - K_{t_0}^{\lambda, \pi}(x_0))}{\log(1/\lambda)} \right) \wedge 1$ where $K_{t_0}^{\lambda, \pi}(x_0) = (2\mathbf{m}_\lambda + 1)^{-1} \left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda X_0 \rfloor + \mathbf{m}_\lambda \rrbracket : \eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1 \right\} \right|$. We work on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$ and set $t_c = t_0 - \tau_{t_0}(x_0)$.

Case 1. If $t_c \geq 1$, we have checked in Step 4 that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1$ for all $i \in (x_0)_\lambda$, whence $K_{t_0}^{\lambda, \pi}(x_0) = 1$ and $Z_{t_0}^{\lambda, \pi}(x_0) = 1$.

Case 2. If now $0 < t_c < 1$, we deduce from Step 3 that

$$\underline{K}_{t_0}^{\lambda, \pi}(x_0) \leq K_{t_0}^{\lambda, \pi}(x_0) \leq \overline{K}_{t_0}^{\lambda, \pi}(x_0)$$

where

$$\begin{aligned}\underline{K}_{t_0}^{\lambda, \pi}(x_0) &= (2\mathbf{m}_\lambda + 1)^{-1} \left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda X_0 \rfloor + \mathbf{m}_\lambda \rrbracket : \eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1 \right\} \right|, \\ \overline{K}_{t_0}^{\lambda, \pi}(x_0) &= (2\mathbf{m}_\lambda + 1)^{-1} \left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda X_0 \rfloor + \mathbf{m}_\lambda \rrbracket : \bar{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1 \right\} \right|.\end{aligned}$$

The random variable $\underline{X}_{t_0}^{\lambda,\pi}(x_0) = (2\mathbf{m}_\lambda + 1)\underline{K}_{t_0}^{\lambda,\pi}(x_0)$ has a binomial distribution with parameters $2\mathbf{m}_\lambda + 1$ and $1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}$. Then, using Bienaymé-Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P} \left[\underline{K}_{t_0}^{\lambda,\pi}(x_0) \leq 1 - \lambda^{t_c - \varepsilon} \right] &= \mathbb{P} \left[\underline{X}_{t_0}^{\lambda,\pi}(x_0) \leq (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c - \varepsilon}) \right] \\
&\leq \mathbb{P} \left[\left| \underline{X}_{t_0}^{\lambda,\pi}(x_0) - (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}) \right| \geq (2\mathbf{m}_\lambda + 1) (\lambda^{t_c - \varepsilon} - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}) \right] \\
&\leq \frac{(2\mathbf{m}_\lambda + 1) (1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}) \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}}{(2\mathbf{m}_\lambda + 1)^2 (\lambda^{t_c - \varepsilon} - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}})^2} \\
&= \frac{1 - \lambda^{t_c - \mathbf{v}_{\lambda,\pi}}}{(2\mathbf{m}_\lambda + 1) \lambda^{t_c - \mathbf{v}_{\lambda,\pi}} (\lambda^{\mathbf{v}_{\lambda,\pi} - \varepsilon} - 1)^2} \simeq \frac{1}{2\mathbf{m}_\lambda \lambda^{t_c - 2\varepsilon + \mathbf{v}_{\lambda,\pi}}} \\
&\leq \frac{1}{2\mathbf{m}_\lambda \lambda^{t_c - \varepsilon}} \text{ (because } 0 < \mathbf{v}_{\lambda,\pi} < \alpha_0 < \varepsilon \text{)} \\
&\leq \delta.
\end{aligned}$$

By the same way, since $\overline{X}_{t_0}^{\lambda,\pi}(x_0) = (2\mathbf{m}_\lambda + 1)\overline{K}_{t_0}^{\lambda,\pi}(x_0)$ has a binomial distribution with parameters $2\mathbf{m}_\lambda + 1$ and $1 - \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}$,

$$\begin{aligned}
\mathbb{P} \left[\overline{K}_{t_0}^{\lambda,\pi}(x_0) \geq 1 - \lambda^{t_c + \varepsilon} \right] &= \mathbb{P} \left[\overline{X}_{t_0}^{\lambda,\pi}(x_0) \geq (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c + \varepsilon}) \right] \\
&\leq \mathbb{P} \left[\left| \overline{X}_{t_0}^{\lambda,\pi}(x_0) - (2\mathbf{m}_\lambda + 1)(1 - \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}) \right| \geq (2\mathbf{m}_\lambda + 1) (\lambda^{t_c + \mathbf{v}_{\lambda,\pi}} - \lambda^{t_c + \varepsilon}) \right] \\
&\leq \frac{(2\mathbf{m}_\lambda + 1) (1 - \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}) \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}}{(2\mathbf{m}_\lambda + 1)^2 (\lambda^{t_c + \mathbf{v}_{\lambda,\pi}} - \lambda^{t_c + \varepsilon})^2} \simeq \frac{1}{2\mathbf{m}_\lambda \lambda^{t_c + \mathbf{v}_{\lambda,\pi}}} \leq \delta.
\end{aligned}$$

All this implies that,

$$\mathbb{P} \left[K_{t_0}^{\lambda,\pi}(x_0) \in (1 - \lambda^{t_c - \varepsilon}, 1 - \lambda^{t_c + \varepsilon}) \right] \geq 1 - c\delta,$$

for some constant $c > 0$, whence

$$\mathbb{P} \left[Z_{t_0}^{\lambda,\pi}(x_0) \in (t_c - \varepsilon, t_c + \varepsilon) \right] \geq 1 - c\delta.$$

This is nothing but the goal, since $Z_{t_0}(x_0) = t_0 - \tau_{t_0}(x_0) = t_c$ as soon as $Z_{t_0}(x_0) < 1$.

Step 6. It remains to prove (a). On $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we check that

(i) If $Z_{t_0}(x_0) < 1$, then $D_{t_0}(x_0) = \{x_0\}$ and $C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}, [\mathbf{n}_\lambda x_0]) \subset (x_0)_\lambda$ (see Step 4 above), whence

$$D_{t_0}^{\lambda,\pi}(x_0) \subset [x_0 - \mathbf{m}_\lambda / \mathbf{n}_\lambda, x_0 + \mathbf{m}_\lambda / \mathbf{n}_\lambda].$$

We deduce that

$$\delta(D_{t_0}^{\lambda,\pi}(x_0), D_{t_0}(x_0)) \leq 2\mathbf{m}_\lambda / \mathbf{n}_\lambda.$$

(ii) If $Z_{t_0}(x_0) = 1$ and $D_{t_0}(x_0) = [a, b]$, for some $a, b \in \chi_{t_0}$, then

- for all $i \in \llbracket [\mathbf{n}_\lambda a] + \mathbf{m}_\lambda + 2\mathbf{k}_{\lambda,\pi}, [\mathbf{n}_\lambda b] - \mathbf{m}_\lambda - 2\mathbf{k}_{\lambda,\pi} \rrbracket \setminus \left(\cup_{x \in \mathcal{B}_M^D} [x]_{\lambda,\pi} \right)$, $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) = 1$.
Indeed, there is no burning tree in $\llbracket [\mathbf{n}_\lambda a] + \mathbf{k}_{\lambda,\pi}, [\mathbf{n}_\lambda b] - \mathbf{k}_{\lambda,\pi} \rrbracket$ at time $\mathbf{a}_\lambda t_0$ (use a very similar result as in Lemma 8.6). Next, by construction, $Z_{t_0}(y) = 1$ for all $y \in (a, b)$ whence $\tau_{t_0}(y) \leq t_0 - 1$. Using $\Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we deduce that $\tau_{t_0}(y) \leq t_0 - 1 - 3\alpha_0$. Using finally Lemma 8.5 and $\Omega_3^S(\lambda, \pi)$, we deduce the claim;
- for all $x \in \mathcal{B}_M^D \cap (a, b)$, and all $i \in [x]_{\lambda,\pi}$, $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) = 1$. Indeed, on $\Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we have $\tilde{H}_{t_0-}(x) = 0$ whence $\tau_{t_0}(x_0) \leq t_0 - 1 - 3\alpha_0$. We deduce that no match falling outside $[x]_{\lambda,\pi}$ affect this zone during the time interval $[\mathbf{a}_\lambda(t_0 - 1 - \alpha_0), \mathbf{a}_\lambda t_0]$ and conclude by distinguishing several cases, as in Step 3 in the proof of Lemma 8.12;

- if $a \in \chi_{t_0}^+ \cup \chi_{t_0}^-$ there is $i \in \langle a \rangle_{\lambda, \pi}$ such that $\eta_{\mathbf{a}_{\lambda} t_0}^{\lambda, \pi}(i) = 2$ (thanks to $\Omega_T^{\lambda, \pi}$, since on $\Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we have $|t_0 - s| \geq 3\alpha$ for all $s \in \mathcal{T}_M^D$ whereas if $a \in \chi_{t_0}^0$, there is $i \in (a)_{\lambda}$ such that $\eta_{\mathbf{a}_{\lambda} t_0}^{\lambda, \pi}(i) = 0$ (use similar argument as in Lemma 8.13, observing that $|t_0 - s| \geq 3\alpha$ for all $s \in \mathcal{T}_M^D$). Similar observation of course holds for b ;

so that

$$\begin{aligned} \llbracket \lfloor \mathbf{n}_{\lambda} a \rfloor + \mathbf{m}_{\lambda} + 2\mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_{\lambda} b \rfloor - \mathbf{m}_{\lambda} - 2\mathbf{k}_{\lambda, \pi} \rrbracket &\subset C(\eta_{\mathbf{a}_{\lambda} t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_{\lambda} x_0 \rfloor) \\ &\subset \llbracket \lfloor \mathbf{n}_{\lambda} a \rfloor - \mathbf{m}_{\lambda} - \mathbf{k}_{\lambda, \pi}, \lfloor \mathbf{n}_{\lambda} b \rfloor + \mathbf{m}_{\lambda} + \mathbf{k}_{\lambda, \pi} \rrbracket \end{aligned}$$

and thus

$$\left[a + \frac{\mathbf{m}_{\lambda} + 2\mathbf{k}_{\lambda, \pi}}{\mathbf{n}_{\lambda}}, b - \frac{\mathbf{m}_{\lambda} + 2\mathbf{k}_{\lambda, \pi}}{\mathbf{n}_{\lambda}} \right] \subset D_{t_0}^{\lambda, \pi}(x_0) \subset \left[a - \frac{\mathbf{m}_{\lambda} + 2\mathbf{k}_{\lambda, \pi}}{\mathbf{n}_{\lambda}}, b + \frac{\mathbf{m}_{\lambda} + 2\mathbf{k}_{\lambda, \pi}}{\mathbf{n}_{\lambda}} \right],$$

$$\text{whence } \delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq 4(\mathbf{m}_{\lambda} + \mathbf{k}_{\lambda, \pi})/\mathbf{n}_{\lambda}.$$

Thus, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we always have $\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq 4(\mathbf{m}_{\lambda} + \mathbf{k}_{\lambda, \pi})/\mathbf{n}_{\lambda}$. We conclude that

$$\mathbb{P} \left[\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq \varepsilon \right] \geq \mathbb{P} \left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi) \right] \geq 1 - \delta.$$

This concludes the proof of Theorem 6.1 for $p > 0$. \square

8.6 Cluster size distribution when $p > 0$

The aim of this section is to prove Corollary 2.7 when $p > 0$.

8.6.1 Study of the LFFP(p)

Recall Subsection 2.1 and Definition 2.2.

Definition 8.16. Let $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(p). For all $x \in \mathbb{R}$ and all $t \geq 0$, we define

$$\mathcal{D}_t(x) = [\mathcal{L}_t(x), \mathcal{R}_t(x)]$$

where

$$\begin{aligned} \mathcal{L}_t(x) &= \inf \left\{ y \leq x : \forall (r, v) \in \Lambda_{(x, t)}^p(y, t - p(x - y)), Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0 \right\}, \\ \mathcal{R}_t(x) &= \sup \left\{ y \geq x : \forall (r, v) \in \Lambda_{(x, t)}^p(y, t + p(x - y)), Z_{v-}(r) = 1 \text{ and } H_{v-}(r) = 0 \right\}. \end{aligned}$$

Observe that for all $t \in [0, T]$ and all $x \in \mathbb{R}$,

- $Z_t(x) = 0$ if and only if $\pi_M \left((\mathcal{D}_t(x) \times \mathbb{R}) \cap \Lambda_{(x, t)}^p \right) > 0$;
- $\mathcal{D}_t(x) = \{x\}$ if $t \in [0, 1)$;
- $|\mathcal{D}_t(x)| \leq 2(t - 1)/p$.

Lemma 8.17. Let $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(p) and consider $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ and $(\mathcal{D}_t(x))_{t \geq 0, x \in \mathbb{R}}$ the associated processes. There are some constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$, depending only on p , such that the following estimates hold.

- (i) For any $t \in (1, \infty)$, any $x \in \mathbb{R}$, any $z \in [0, 1)$, $\mathbb{P}[Z_t(x) = z] = 0$.
- (ii) For any $t \in [0, \infty)$, any $B > 0$, any $x \in \mathbb{R}$, $\mathbb{P}[|D_t(x)| = B] = 0$.
- (iii) For all $t \in [0, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \leq c_2 e^{-\kappa_1 B}$.
- (iv) For all $t \in [\frac{11}{8}, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \geq c_1 e^{-\kappa_2 B}$.

(v) For all $t \in [0, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|\mathcal{D}_t(x)| \geq B] \leq c_2 e^{-\kappa_1 B}$.

(vi) For all $t \in [\frac{3}{2}, \infty)$, all $x \in \mathbb{R}$, all $B \in (0, (2t-3)/p)$, $\mathbb{P}[|\mathcal{D}_t(x)| \geq B] \geq c_1 e^{-\kappa_2(B+B^2)}$.

(vii) For all $t \in [(5+p)/2, \infty)$, all $0 \leq a < b < 1$, all $x \in \mathbb{R}$,

$$c_1(b-a) \leq \mathbb{P}[Z_t(x) \in [a, b]] \leq c_2(b-a).$$

Proof. By invariance by translation, it suffices to treat the case $x = 0$.

Point (i). For $t \in [0, 1]$, we have a.s. $Z_t(0) = t$. But for $t > 1$ and $z \in [0, 1]$, $Z_t(0) = z$ implies that a fire has crossed 0 at time $t - z$, so that necessarily $\pi_M(\Lambda_{(0,t)}^p) > 0$, recall Subsection 2.1. This happens with probability 0.

Point (ii). For any $t > 0$, $|D_t(0)|$ is either 0 or of the form $|x - y|$, for some $x, y \in \chi_t$. We easily conclude as previously that for $B > 0$, $\Pr(|D_t(0)| = B) = 0$.

Point (iii). First if $t \in [0, 1)$, we have a.s. $|D_t(0)| = 0$ and the result is obvious. Recall that for (X, τ) a mark of π_M , we have $H_t(X) > 0$ or $Z_t(X) < 1$ for all $t \in [\tau, \tau + 1/2)$ (see the proof of Proposition 3.5-Step 1). This implies that for $t \geq 1$,

$$\begin{aligned} \{|D_t(0)| \geq B\} &\subset \{[0, B/2] \text{ is connected at time } t \text{ or } [-B/2, 0] \text{ is connected at time } t\} \\ &\subset \{\pi_M([0, B/2] \times [t - 1/4, t]) = 0\} \cup \{\pi_M([-B/2, 0] \times [t - 1/4, t]) = 0\}. \end{aligned}$$

Consequently, $\Pr[|D_t(0)| \geq B] \leq 2e^{-B/8}$ as desired.

Point (iv). Fix $B > 0$ and $t \geq 11/8$. Set $\Delta = \frac{3}{16p}$ and $K = \left\lfloor \frac{1}{\Delta} \left(B + \frac{11}{4p} \right) \right\rfloor + 1$. Consider the event $\Omega_{t,B} = \Omega_{t,B}^0 \cap \bigcap_{k=0}^{K-1} \Omega_{t,B,k}$, illustrated by Figure 8, where

- $\Omega_{t,B}^0 = \{\pi_M([-5/(4p), B + 5/(4p)] \times [t - 5/4, t]) = 0\}$;
- for all $k \in [0, K - 1]$, $\Omega_{t,B,k} = \{\pi_M(D_k) = 1\} \cap \{\pi_M(C_k \setminus D_k) = 0\}$ where

$$\begin{aligned} C_k &= \left[-\frac{11}{8p} + k\Delta, -\frac{11}{8p} + (k+1)\Delta \right] \times [t - 11/8, t - 5/4] \\ D_k &= \left[-\frac{11}{8p} + (k + \frac{1}{3})\Delta, -\frac{11}{8p} + (k + \frac{2}{3})\Delta \right] \times [t - 11/8, t - 5/4], \end{aligned}$$

see Figure 9. Observe that $\bigcup_{k=0}^{K-1} C_k \supset [-11/(8p), B + 11/(8p)]$.

We have $\mathbb{P}[\Omega_{t,B}^0] = \exp\left(-\frac{5}{4}(B + \frac{5}{2p})\right)$ whence for all $k \in [0, K - 1]$, $\mathbb{P}[\Omega_{t,B,k}] = \frac{\Delta}{24} \times e^{-\frac{\Delta}{24}} \times e^{-\frac{\Delta}{12}}$. All these events being independent, we conclude that

$$\mathbb{P}[\Omega_{t,B}] = \exp\left(-\frac{5}{4}(B + \frac{5}{2p})\right) \times \left(\frac{\Delta}{24} e^{-\frac{\Delta}{8}}\right)^K \geq c_1 e^{-\kappa_2 B}$$

for some constant c_1 and κ_2 not depending on B . To conclude the proof of (iv), it thus suffices to check that $\Omega_{t,B} \subset \{[0, B] \subset D_t(0)\}$. But on $\Omega_{t,B}$, using the same arguments as in Point (iii), we observe that:

- for (X, τ) a mark of π_M , $H_s^A(X) > 0$ or $Z_s^A(X) < 1$ for all $s \in [\tau, \tau + 3/8]$. Thus, for all $k \in [0, K - 1]$, there is $x \in D_k$ such that $H_s^A(x) > 0$ or $Z_s^A(x) < 1$ for all $s \in [t - 5/4, t - 1]$;
- calling (X_k, τ_k) the mark of π_M in D_k , we have $\tau_k + p(X_{k+1} - X_k) \in [t - 5/4, t - 1]$ and $\tau_k + p(X_k - X_{k-1}) \in [t - 5/4, t - 1]$, see Figure 9. Thus, if the fire starting on X_k at time τ_k is macroscopic, it is (at least) stopped by the marks (X_{k-1}, τ_{k-1}) and (X_{k+1}, τ_{k+1}) and does not affect the zone $[0, B]$ after $t - 1$;

- for (Y, S) a mark of π_M such that $(Y, S) \notin [-11/(8p), B + 11/(8p)] \times [t - 11/8, t]$ and $Y + (t - S)/p \in [0, B]$, then there exists $k \in \llbracket 0, K - 1 \rrbracket$ such that

$$Y + \frac{t - 11/8 - S}{p} \in \left[-\frac{11}{8p} + (k - \frac{1}{3})\Delta, -\frac{11}{8p} + (k + \frac{2}{3})\Delta \right].$$

We immediately conclude that $S + p(X_{k+1} - Y) \in [t - 5/4, t - 1]$. Thus, the right front of (Y, S) is stopped by the match (X_{k+1}, τ_{k+1}) and does not affect the zone $[0, B]$ after $t - 1$;

- for (Y, S) a mark of π_M such that $(Y, S) \notin [-11/(8p), B + 11/(8p)] \times [t - 11/8, t]$ and $Y - (t - S)/p \in [0, B]$, we prove as above that the left front of (Y, S) is stopped by such a match (X_{k-1}, τ_{k-1}) and does not affect the zone $[0, B]$ after $t - 1$;
- by construction, the other fires may not affect the zone $[-11/(8p), B + 11/(8p)]$ during the time interval $[t - 1, t]$.

As a conclusion, the zone $[0, B]$ is not affected by any fire during $[t - 1, t]$. Since the length of this time interval is greater than 1, we deduce that for all $x \in [0, B]$, $Z_t(x) = \min(Z_{t-1}(x) + 1, 1) = 1$ and $H_t(x) = \max(H_{t-1}(x) - 1, 0) = 0$, whence $[0, B] \subset D_t(0)$.

Point (v) First if $t \in [0, 1)$, we have a.s. $|\mathcal{D}_t(0)| = 0$ and the result is obvious. If $t \geq 1$ and $B > 2(t - 1)/p$,

$$\mathbb{P}[|\mathcal{D}_t(0)| \geq B] = 0.$$

Recall that for (X, τ) a mark of π_M , we have $H_t(X) > 0$ or $Z_t(X) < 1$ for all $t \in [\tau, \tau + 1/2]$ (see the proof of Proposition 3.5-Step 1). This implies that for $t \geq 1$ and $B \in (0, 2(t - 1)/p)$,

$$\begin{aligned} \{|\mathcal{D}_t(0)| \geq B\} &\subset \{[0, B/2] \subset [0, \mathcal{D}_t(x)] \text{ or } [-B/2, 0] \subset [\mathcal{L}_t(x), 0]\} \\ &\subset \left\{ \pi_M \left(\left\{ (r, v) \in \Lambda_{(0,s)}^p(B/2, s - pB/2) : s \in [t - 1/4, t] \right\} \right) = 0 \right\} \\ &\quad \cup \left\{ \pi_M \left(\left\{ (r, v) \in \Lambda_{(0,s)}^p(-B/2, s - pB/2) : s \in [t - 1/4, t] \right\} \right) = 0 \right\}. \end{aligned}$$

Consequently, $\mathbb{P}[|\mathcal{D}_t(0)| \geq B] \leq 2e^{-B/8}$, as desired.

Point (vi) Let $t \geq 3/2$ and $B \in (0, (2t - 3)/p)$. From Point (iv), using space/time stationarity, we define an event $\tilde{\Omega}_{t,B}$, depending on the Poisson measure $\pi_M(dx, ds)$ restricted to $[-B/2 - 11/(8p), B/2 + 11/(8p)] \times [t - pB/2 - 3/2, t - pB/2]$, on which $D_{t-pB/2}(0) \supset [-B/2, B/2]$. Next consider the event

$$\tilde{\Omega}_{t,B}^0 = \{\pi_M([-B/2, B/2] \times [t - pB/2, t]) = 0\}.$$

We have $\mathbb{P}[\tilde{\Omega}_{t,B}^0] = e^{-pB^2/2}$.

The events $\tilde{\Omega}_{t,B}$ and $\tilde{\Omega}_{t,B}^0$ are independent, thus we have, recalling point (iv)

$$\mathbb{P}[\tilde{\Omega}_{t,B} \cap \tilde{\Omega}_{t,B}^0] = \mathbb{P}[\tilde{\Omega}_{t,B}] \times \mathbb{P}[\tilde{\Omega}_{t,B}^0] \geq c_1 e^{-\kappa_2(B+B^2)}.$$

Finally, we observe that for $(X, t - pB/2)$ a fire at time $t - pB/2$ with, for example, $X < -B/2$, we have, by construction, $X + (t - (t - pB/2))/p < 0$. Thus,

$$\tilde{\Omega}_{t,B} \cap \tilde{\Omega}_{t,B}^0 \subset \{|\mathcal{D}_t(0)| \geq B\}.$$

This concludes the point.

Point (vii) For $0 \leq a \leq b < 1$ and $t \geq 1$, we have $Z_t(0) \in [a, b]$ if and only if there is $\tau \in [t - b, t - a]$ such that $Z_\tau(0) = 0$. And this happens if and only if

$$X_{t,a,b} := \int_{t-b}^{t-a} \int_{\mathbb{R}} \mathbf{1}_{\{(y, s-p|x-y|) \in \mathcal{D}_{s-(0)} \times [0, s]\}} \pi_M(dy, ds) \geq 1.$$

We deduce that

$$\mathbb{P}[Z_t(0) \in [a, b]] = \mathbb{P}[X_{t,a,b} \geq 1] \leq \mathbb{E}[X_{t,a,b}] = \int_{t-b}^{t-a} \mathbb{E}[|\mathcal{D}_s(0)|] ds \leq C(b - a),$$

where we used Point (v) for the last inequality.

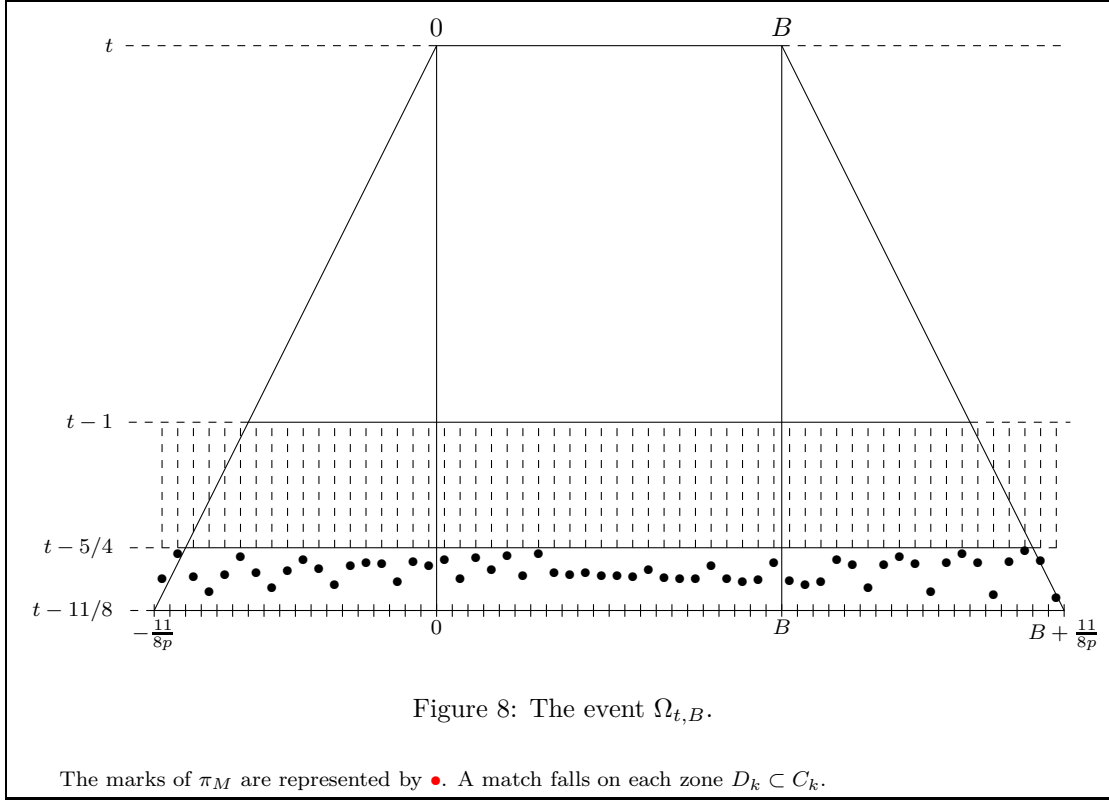
Next, we have $\{\pi_M(\mathcal{D}_{t-b}(0) \times [t-b, t-a]) \geq 1\} \subset \{X_{t,a,b} \geq 1\}$: it suffices to note that a.s.,

$$\begin{aligned} \{X_{t,a,b} = 0\} &\subset \{X_{t,a,b} = 0, \mathcal{D}_{t-b}(0) \subset \mathcal{D}_s(0) \text{ for all } s \in [t-b, t-a]\} \\ &\subset \{\pi_M(\mathcal{D}_{t-b}(0) \times [t-b, t-a]) = 0\}. \end{aligned}$$

Since now $\mathcal{D}_{t-b}(0)$ is independent of $\pi_M(dx, ds)$ restricted to $\mathbb{R} \times (t-b, \infty)$, we deduce that for $t \geq (5+p)/2$

$$\begin{aligned} \mathbb{P}[Z_t(0) \in [a, b]] &\geq \mathbb{P}[\pi_M(\mathcal{D}_{t-b}(0) \times [t-b, t-a]) \geq 1] \\ &\geq \mathbb{P}[|\mathcal{D}_{t-b}(0)| \geq 1] (1 - e^{-(b-a)}) \\ &\geq c(1 - e^{-(b-a)}), \end{aligned}$$

where we used Point (vi) (here $t-b \geq 3/2$ and $(2t-3)/p \geq 1$) to get the last inequality. This concludes the proof, since $1 - e^{-x} \geq x/2$ for all $x \in [0, 1]$. \square



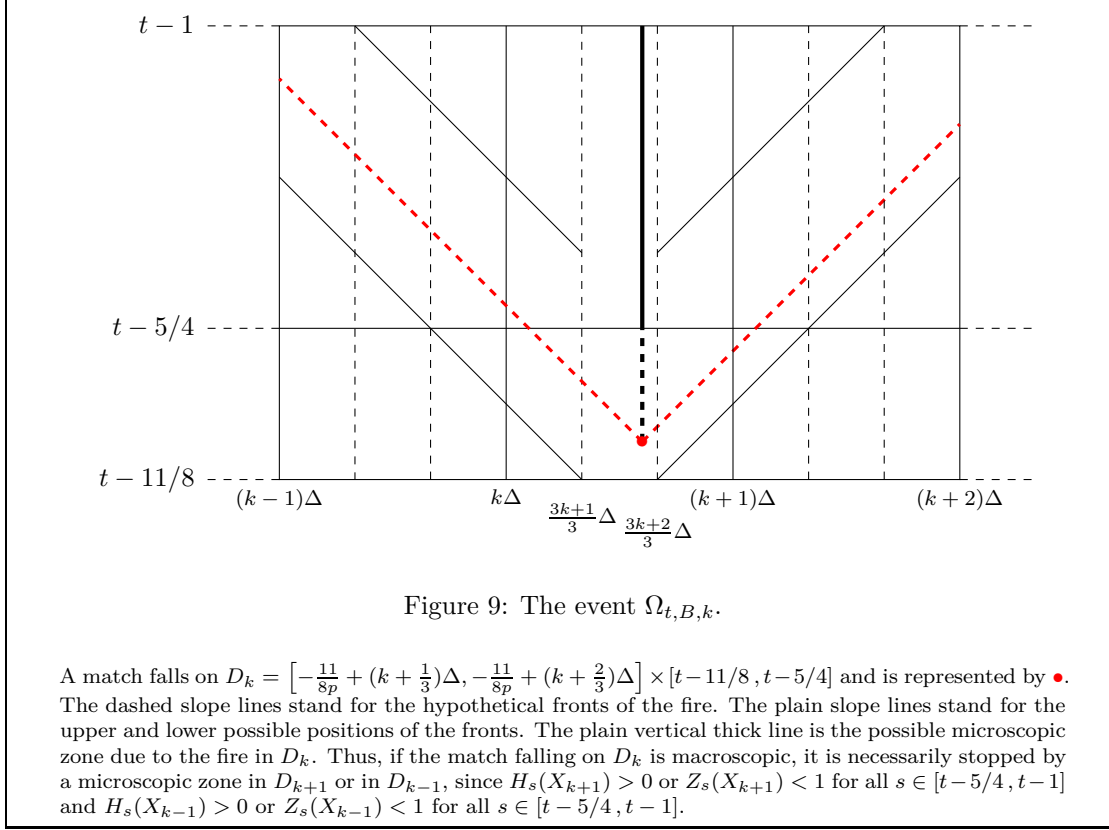
8.6.2 Proof of Corollary 2.7 when $p > 0$

We finally give the

Proof of Corollary 2.7 when $p > 0$. For each $\lambda \in (0, 1)$ and each $\pi \geq 1$, consider a (λ, π) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$. Let also $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(p) and consider the corresponding process $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$.

Point (b). Using Lemma 8.17-(iii)-(iv) and recalling that $|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)|/\mathbf{n}_\lambda = |D_t^{\lambda, \pi}(0)|$, it suffices to check that for all $t \geq 3/2$ and all $B > 0$, when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$,

$$\lim_{\lambda, \pi} \mathbb{P}[|D_t^{\lambda, \pi}(0)| \geq B] = \mathbb{P}[|D_t(0)| \geq B].$$



This follows from Theorem 2.5-2, which implies that $|D_t^{\lambda,\pi}(0)|$ goes in law to $|D_t(0)|$ and from Lemma 8.17-(ii).

Point (a). Due to Lemma 8.17-(v) we only need that for all $0 < a < b < 1$, all $t \geq (5 + p)/2$, when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$,

$$\lim_{\lambda,\pi} \mathbb{P} \left[|C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}, 0)| \in [\lambda^{-a}, \lambda^{-b}] \right] = \mathbb{P} [Z_t(0) \in [a, b]].$$

But using Theorem 2.5-3 and Lemma 8.17-(i), we know that

$$\lim_{\lambda,\pi} \mathbb{P} \left[\frac{\log(|C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}, 0)| \geq 1\}} \in [a, b] \right] = \mathbb{P} [Z_t(0) \in [a, b]]$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$. One immediately concludes. \square

9 Convergence in the regime $\mathcal{R}(0)$

The aim of this section is to prove Theorem 6.1 when $p = 0$ and this will conclude the proof of Theorem 2.5.

In the whole section, we fix the parameters $A > 0$ and $T > 2$. We omit the subscript/superscript A in the whole proof. The proof follows the ideas of the Section 8.

We recall that $\mathbf{a}_\lambda = \log(1/\lambda)$, $\mathbf{n}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda) \rfloor$, $\mathbf{m}_\lambda = \lfloor 1/(\lambda \mathbf{a}_\lambda^2) \rfloor$, $\varepsilon_\lambda = 1/\mathbf{a}_\lambda^3$. We set as usual $A_\lambda = \lfloor \mathbf{n}_\lambda A \rfloor$ and $I_A^\lambda = \llbracket -A_\lambda, A_\lambda \rrbracket$. For $i \in \mathbb{Z}$, we set $i_\lambda = \lfloor i/\mathbf{n}_\lambda, (i+1)/\mathbf{n}_\lambda \rfloor$. For $[a, b]$ an interval of $[-A, A]$ and $\lambda \in (0, 1)$, we recall, assuming that $-A < a < b < A$, that

$$\begin{aligned} [a, b]_\lambda &= \llbracket \lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda + 1, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1 \rrbracket \subset \mathbb{Z}, \\ [-A, b]_\lambda &= \llbracket -A_\lambda, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1 \rrbracket \subset \mathbb{Z}, \\ [a, A]_\lambda &= \llbracket \lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda + 1, A_\lambda \rrbracket \subset \mathbb{Z}. \end{aligned}$$

For $\lambda \in (0, 1)$ and $\pi \geq 1$, we recall that

$$\varkappa_{\lambda, \pi} = \frac{2\mathbf{n}_\lambda A}{\mathbf{a}_\lambda \pi} + \varepsilon_\lambda.$$

For $x \in (-A, A)$, $\lambda \in (0, 1)$ and $\pi \geq 1$, we also recall that

$$(x)_\lambda = \llbracket \lfloor \mathbf{n}_\lambda x \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x \rfloor + \mathbf{m}_\lambda \rrbracket \subset \mathbb{Z}.$$

9.1 Occupation of vacant zone

For simplicity, we recall Lemma 8.1.

Lemma 9.1. *Consider a family of i.i.d. Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$. Let $a < b$.*

1. *For $t < 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\forall i \in \llbracket \lfloor a\mathbf{m}_\lambda \rfloor, \lfloor b\mathbf{m}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$;*
2. *For $t \geq 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\forall i \in \llbracket \lfloor a\mathbf{m}_\lambda \rfloor, \lfloor b\mathbf{m}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
3. *For $t < 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\forall i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$;*
4. *For $t \geq 1$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\forall i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
5. *For $t > 0$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\exists i \in \llbracket \lfloor a\mathbf{n}_\lambda \rfloor, \lfloor b\mathbf{n}_\lambda \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$;*
6. *For $t > 0$ and $\delta > 0$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\forall i \in \llbracket -\lfloor \lambda^{-(t+\delta)} \rfloor, \lfloor \lambda^{-(t+\delta)} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 0$;*
7. *For $t > 0$ and $\delta > 0$, $\lim_{\lambda \rightarrow 0} \mathbb{P}[\forall i \in \llbracket -\lfloor \lambda^{-(t-\delta)} \rfloor, \lfloor \lambda^{-(t-\delta)} \rfloor \rrbracket, N_{\mathbf{a}_\lambda t}^S(i) > 0] = 1$.*

9.2 Height of the barrier

We describe here the time needed for a destroyed microscopic cluster to be regenerated. Roughly, we assume that the zone $(x_1)_\lambda$ around $\lfloor \mathbf{n}_\lambda x_1 \rfloor$, for some $x_1 \in [-A, A]$, has been made vacant at some time $\mathbf{a}_\lambda t_0$. Then we consider the situation where a match falls on $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at some time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + 1))$ and we compute the delay needed for the destroyed cluster to be fully regenerated. As in Subsection 8.2, we have to distinguish the cases $t_0 = 0$ and $t_0 > 1$.

Lemma 9.2. *Consider two Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all these processes being independent. Consider also $\mathcal{M} := ((x_0, t_0), (x_1, t_1))$ with $x_0, x_1 \in (-A, A)$, $t_0 \in \{0\} \cup (1, \infty)$ and $t_1 \in (t_0, t_0 + 1)$. For $i \in I_A^\lambda$ and $t \geq 0$, we consider the process*

$$\begin{aligned} \zeta_t^{\lambda, \pi, \mathcal{M}}(i) &= (1 + \mathbf{1}_{\{t \geq \mathbf{a}_\lambda t_0, i = \lfloor \mathbf{n}_\lambda x_0 \rfloor\}}) \times \mathbf{1}_{\{t_0 > 1\}} \\ &\quad + \mathbf{1}_{\{t \geq \mathbf{a}_\lambda t_1, i = \lfloor \mathbf{n}_\lambda x_1 \rfloor, \zeta_{\mathbf{a}_\lambda t_1 -}^{\lambda, \pi, \mathcal{M}}(\lfloor \mathbf{n}_\lambda x_1 \rfloor) = 1\}} + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 0\}} dN_s^S(i) \\ &\quad + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i+1) = 2, \zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 1\}} dN_s^P(i+1) \\ &\quad + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i-1) = 2, \zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 1\}} dN_s^P(i-1) \\ &\quad - 2 \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{M}}(i) = 2\}} dN_s^P(i) \end{aligned}$$

with the convention $\zeta_t^{\lambda, \pi, \mathcal{M}}(\lfloor \mathbf{n}_\lambda A \rfloor + 1) = \zeta_t^{\lambda, \pi, \mathcal{M}}(-\lfloor \mathbf{n}_\lambda A \rfloor - 1) = 0$ for all $t \in [0, \infty)$.

Using the Poisson processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, consider the burning times $(T_i^1)_{i \in \mathbb{Z}}$ of the propagation processes ignited at (x_1, t_1) , recall Definition 4.6, and define the destroyed cluster due to the match falling in $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at time $\mathbf{a}_\lambda t_1$, recall Definition 4.8,

$$C^P((\zeta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_1, t_1)) := \llbracket \lfloor \mathbf{n}_\lambda x_1 \rfloor + i^g, \lfloor \mathbf{n}_\lambda x_1 \rfloor + i^d \rrbracket.$$

We finally define the time needed for $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_1, t_1))$ to become again occupied

$$\Theta_{\mathcal{M}}^{\lambda,\pi} := \inf \left\{ t > t_1 : \forall i \in C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_1, t_1)), \zeta_{\mathbf{a}_\lambda t}^{\lambda,\pi,\mathcal{M}}(i) = 1 \right\}.$$

For all $\delta > 0$, there holds that,

$$\lim_{\lambda, \pi} \mathbb{P} \left[\left| \Theta_{\mathcal{M}}^{\lambda,\pi} - (t_1 - t_0) \right| \geq \delta \right] = 0$$

when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

The process $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}$ defined in Lemma 9.2 is closely related to the process defined in Lemma 8.2. If $t_0 = 0$, then the process starts from a vacant initial situation and a match falls on $[\mathbf{n}_\lambda x_1]$ at time $\mathbf{a}_\lambda t_1$. It does not depend on $x_0 \in \mathbb{R}$. Since $0 < t_1 < 1$, the zone $(x_1)_\lambda$ is not completely filled at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$, see Lemma 9.1-1 (using space stationarity). The process is then governed by the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ and the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ with the same rules as the (λ, π) -FFP. As seen in **Micro**(0) in Subsection 4.4, the fire is extinguish at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$.

If $t_0 > 1$, then the process starts at time 0 from an occupied initial situation, nothing happens until a match falls in $[\mathbf{n}_\lambda x_0] \in I_A^\lambda$ at time $\mathbf{a}_\lambda t_0$. Two fires start: one goes to the left and one goes to the right. Thus, on $\Omega_{\lambda,\pi}^{P,2A,2A}(x_0, t_0)$, recall Definition 4.7, each site of I_A^λ burns and extinguishes before $\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda,\pi})$, recall **Macro**(0) in Subsection 4.4. Hence, the zone $(x_1)_\lambda$ is not completely filled when the match falls on $[\mathbf{n}_\lambda x_1]$ at time $\mathbf{a}_\lambda t_1$, see Lemma 9.1-1, because $\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda,\pi}) < \mathbf{a}_\lambda t_1 < \mathbf{a}_\lambda(t_0 + 1)$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

Proof. The proof is very similar to the proof of Lemma 8.2. We first define the simplest process with an instantaneous propagation: if a match falls in a cluster, it destroys instantaneously the entire connected component. Secondly, we flank the killed cluster $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (x_1, t_1))$ to estimate the time needed to become again occupied.

Without loss of generality, we assume that $x_1 = 0$ and $x_0 \in [-A, A]$ (using space stationarity).

Step 1. Let $\tau_0 < \tau_1 < \tau_0 + 1$ be fixed. Put $\vartheta_{\tau_0,t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_0+t)}^S(i) - N_{\mathbf{a}_\lambda\tau_0}^S(i), 1)$ and $\vartheta_{\tau_1,t}^\lambda(i) = \min(N_{\mathbf{a}_\lambda(\tau_1+t)}^S(i) - N_{\mathbf{a}_\lambda\tau_1}^S(i), 1)$ for all $t > 0$ and all $i \in \mathbb{Z}$. We define the time needed for the destroyed cluster to be fully regenerated

$$\Xi_{\tau_0,\tau_1}^\lambda = \inf \left\{ t > 0 : \forall i \in C(\vartheta_{\tau_0,\tau_1-\tau_0}^\lambda, 0), \vartheta_{\tau_1,t}^\lambda(i) = 1 \right\}.$$

Then for all $\delta > 0$,

$$\lim_{\lambda \rightarrow 0} \mathbb{P} \left[|\Xi_{\tau_0,\tau_1}^\lambda - (\tau_1 - \tau_0)| \geq \delta \right] = 0.$$

This has been checked in Step 1 in the proof of Lemma 8.2.

Step 2. Assume $t_0 = 0$. In that case, the process does not depend on x_0 . Consider the event $\Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1)$, recall Definition 4.7. We define

$$\begin{aligned} \tilde{\Omega}_{\lambda,\pi}^{P,A,\mathcal{M}} &= \Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1) \cap \{ \exists i_1 \in \llbracket -\mathbf{m}_\lambda, 0 \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})}^S(i_1) = 0 \} \\ &\quad \cap \{ \exists i_2 \in \llbracket 0, \mathbf{m}_\lambda \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})}^S(i_2) = 0 \}. \end{aligned}$$

Lemma 4.3 together with Lemma 9.1-1 show that $\mathbb{P} \left[\tilde{\Omega}_{\lambda,\pi}^{P,A,\mathcal{M}} \right]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$ (because $t_1 + \varkappa_{\lambda,\pi} < (t_1 + 1)/2 < 1$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$).

Next, on $\tilde{\Omega}_{\lambda,\pi}^{P,A,\mathcal{M}}(0, t_1)$, there holds that

$$C(\vartheta_{0,t_1+\varkappa_{\lambda,\pi}}^\lambda, 0) := \llbracket C^-, C^+ \rrbracket \subset \llbracket i_1, i_2 \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket.$$

Since, by definition, no seed falls on C^+ and on C^- until $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$ and since we start from a vacant initial situation, we also deduce that

$$\zeta_t^{\lambda,\pi,\mathcal{M}}(C^-) = \zeta_t^{\lambda,\pi,\mathcal{M}}(C^+) = 0$$

for all $t \in [0, \mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})] \supset [\mathbf{a}_\lambda t_1, \mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})]$. As seen in **Micro**(0) in Subsection 4.4, the match falling on 0 at time $\mathbf{a}_\lambda t_1$ destroys exactly the zone $C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ and

$$C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) \subset \llbracket C^-, C^+ \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket$$

with $\zeta_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})}^{\lambda, \pi, \mathcal{M}}(i) \leq 1$ for all $i \in \mathbb{Z}$ (the fire is extinguished at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})$).

Since $C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ clearly contains $C(\vartheta_{0, t_1}^\lambda, 0)$, we deduce that, on $\tilde{\Omega}_{\lambda, \pi}^{P, A, \mathcal{M}}$,

$$t_1 + \Xi_{0, t_1}^\lambda \leq t_1 + \Theta_{\mathcal{M}}^{\lambda, \pi} \leq t_1 + \varkappa_{\lambda, \pi} + \Xi_{0, t_1 + \varkappa_{\lambda, \pi}}^\lambda.$$

Remark now that the function $t \mapsto t + \Xi_{0, t}^\lambda$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1 + \Theta_{\mathcal{M}}^{\lambda, \pi} \xrightarrow[\lambda, \pi]{\mathbb{P}} 2t_1$$

in probability, whence for all $\delta > 0$ and all $\varepsilon > 0$, there holds that $\mathbb{P} \left[\left| \Theta_{\mathcal{M}}^{\lambda, \pi} - t_1 \right| \geq \delta \right] < \varepsilon$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

Step 3. Assume now $t_0 > 1$. We may and will assume $x_0 \in (-A, 0)$, by symmetry.

Consider the events $\Omega_{\lambda, \pi}^{P, 2A, 2A}(x_0, t_0)$ and $\Omega_{\lambda, \pi}^{P, 2A, 2A}(0, t_1)$, recall Definition 4.7. We define

$$\begin{aligned} \tilde{\Omega}_{\lambda, \pi}^{P, A, \mathcal{M}} &:= \Omega_{\lambda, \pi}^{P, 2A, 2A}(0, t_1) \cap \Omega_{\lambda, \pi}^{P, 2A, 2A}(x_0, t_0) \\ &\cap \{ \exists i_1 \in \llbracket -\mathbf{m}_\lambda, 0 \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})}^S(i_1) - N_{\mathbf{a}_\lambda t_0}^S(i_1) = 0 \} \\ &\cap \{ \exists i_2 \in \llbracket 0, \mathbf{m}_\lambda \rrbracket, N_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})}^S(i_2) - N_{\mathbf{a}_\lambda t_0}^S(i_2) = 0 \}. \end{aligned}$$

Lemma 4.3 together with Lemma 9.1-1 directly imply that $\mathbb{P} \left[\tilde{\Omega}_{\lambda, \pi}^{P, A, \mathcal{M}} \right]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$ (because $t_1 + \varkappa_{\lambda, \pi} - t_0 < (t_1 - t_0 + 1)/2 < 1$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$).

First, since the sites $\lfloor \mathbf{n}_\lambda A \rfloor + 1$ and $-\lfloor \mathbf{n}_\lambda A \rfloor - 1$ remain vacant all the time and since I_A^λ is completely occupied at time $\mathbf{a}_\lambda t_0$, on $\Omega_{\lambda, \pi}^{P, 2A, 2A}(x_0, t_0)$, as seen in **Macro**(0) in Subsection 4.4, the match falling on $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$ destroys each site of I_A^λ during the time interval $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \varkappa_{\lambda, \pi})]$. Furthermore, there is no more burning tree in I_A^λ at time $\mathbf{a}_\lambda(t_0 + \varkappa_{\lambda, \pi})$.

Next, on $\tilde{\Omega}_{\lambda, \pi}^{P, A, \mathcal{M}}$, since no seed falls on i_1 and i_2 during the time interval $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})]$, we clearly have

$$C(\vartheta_{t_0, t_1 + \varkappa_{\lambda, \pi}}^\lambda, 0) := \llbracket C^-, C^+ \rrbracket \subset \llbracket i_1, i_2 \rrbracket \subset \llbracket -\mathbf{m}_\lambda, \mathbf{m}_\lambda \rrbracket.$$

Since, by definition, no seed falls on C^- and on C^+ during $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})]$ and since C^- and C^+ are made vacant during the time interval $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + \varkappa_{\lambda, \pi})]$, we deduce that

$$\zeta_{\mathbf{a}_\lambda t}^{\lambda, \pi, \mathcal{M}}(C^-) = \zeta_{\mathbf{a}_\lambda t}^{\lambda, \pi, \mathcal{M}}(C^+) = 0 \text{ for all } t \in [t_1, t_1 + \varkappa_{\lambda, \pi}].$$

Hence, as seen in **Micro**(0) in Subsection 4.4, the match falling on 0 at time $\mathbf{a}_\lambda t_1$ destroys exactly the zone $C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) \subset \llbracket C^-, C^+ \rrbracket \subset \llbracket i_1, i_2 \rrbracket$.

To summarize, since $C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$ clearly contains $C(\vartheta_{t_0 + \varkappa_{\lambda, \pi}, t_1}^\lambda, 0)$, on $\tilde{\Omega}_{\lambda, \pi}^{P, A, \mathcal{M}}$, we have

$$C(\vartheta_{t_0 + \varkappa_{\lambda, \pi}, t_1}^\lambda, 0) \subset C^P((\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1)) \subset C(\vartheta_{t_0, t_1 + \varkappa_{\lambda, \pi}}^\lambda, 0) \subset \llbracket i_1, i_2 \rrbracket$$

with additionally $\zeta_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})}^{\lambda, \pi, \mathcal{M}}(i) \leq 1$ for all $i \in I_A^\lambda$.

We deduce that, on $\tilde{\Omega}_{\lambda, \pi}^{P, A, \mathcal{M}}$ and for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$,

$$t_1 + \Xi_{t_0 + \varkappa_{\lambda, \pi}, t_1}^\lambda \leq t_1 + \Theta_{\mathcal{M}}^{\lambda, \pi} \leq t_1 + \varkappa_{\lambda, \pi} + \Xi_{t_0, t_1 + \varkappa_{\lambda, \pi}}^\lambda.$$

Then, one easily concludes. The function $s \mapsto t_1 + \Xi_{t_0 + s, t_1}^\lambda$ is a.s. non increasing and right-continuous, while the function $s \mapsto t_1 + s + \Xi_{t_0, t_1 + s}^\lambda$ is a.s. non decreasing and right-continuous. We thus deduce from Step 1 that

$$t_1 + \Theta_{\mathcal{M}}^{\lambda, \pi} \xrightarrow[\lambda, \pi]{\mathbb{P}} 2t_1 - t_0,$$

as desired. \square

9.3 Persistent effect of microscopic fires

Here we study the effect of microscopic fires. First, they produce a barrier, and then, if there are alternatively macroscopic fires on the left and right, they still have an effect. This phenomenon is illustrated on Figure 10 in the case of the limit process.

We say that $\mathcal{P} = (\varepsilon; (x_0, t_0), (x_1, t_1), \dots, (x_K, t_K))$ satisfies (PP) if

1. $K \geq 2$ and $\varepsilon \in \{-1, 1\}$;
2. $t_0 \in \{0\} \cup (1, \infty)$ and $t_0 < t_1 < t_2 < \dots < t_K$;
3. for all $k = 0, \dots, K-1$, $t_{k+1} - t_k < 1$;
4. $t_2 - t_0 > 1$ and for all $k = 2, \dots, K-2$, $t_{k+2} - t_k > 1$;
5. for all $k = 0, \dots, K$, $x_k \in (-A, A)$ and for all $k = 2, \dots, K$, $\varepsilon_k(x_k - x_1) > 0$, where we set $\varepsilon_k = (-1)^k \varepsilon$.

Let \mathcal{P} satisfy (PP) . Consider two Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective rates 1 and π , all these processes being independent. We define the process $(\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \geq 0, i \in I_A^\lambda}$ as follows

$$\begin{aligned} \zeta_t^{\lambda, \pi, \mathcal{P}}(i) = & (1 + \mathbf{1}_{\{i = \lfloor \mathbf{n}_\lambda x_0 \rfloor, t \geq \mathbf{a}_\lambda t_0\}}) \mathbf{1}_{\{t_0 \geq 1\}} + \mathbf{1}_{\{i = \lfloor \mathbf{n}_\lambda x_1 \rfloor, t \geq \mathbf{a}_\lambda t_1, \zeta_{\mathbf{a}_\lambda t_1 -}^{\lambda, \pi, \mathcal{P}}(\lfloor \mathbf{n}_\lambda x_1 \rfloor) = 1\}} \\ & + \sum_{k=2}^K \mathbf{1}_{\{i = \lfloor \mathbf{n}_\lambda x_k \rfloor, t \geq \mathbf{a}_\lambda t_k, \zeta_{\mathbf{a}_\lambda t_k -}^{\lambda, \pi, \mathcal{P}}(\lfloor \mathbf{n}_\lambda x_k \rfloor) = 1\}} \\ & + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{P}}(i) = 0\}} dN_s^S(i) \\ & + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{P}}(i-1) = 2, \zeta_{s-}^{\lambda, \pi, \mathcal{P}}(i) = 1\}} dN_s^P(i-1) + \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{P}}(i+1) = 2, \zeta_{s-}^{\lambda, \pi, \mathcal{P}}(i) = 1\}} dN_s^P(i+1) \\ & - 2 \int_0^t \mathbf{1}_{\{\zeta_{s-}^{\lambda, \pi, \mathcal{P}}(i) = 2\}} dN_s^P(i) \end{aligned}$$

with the convention $\zeta_t^{\lambda, \pi, \mathcal{P}}(\lfloor \mathbf{n}_\lambda A \rfloor + 1) = \zeta_t^{\lambda, \pi, \mathcal{P}}(-\lfloor \mathbf{n}_\lambda A \rfloor - 1) = 0$ for all $t \in [0, \infty)$.

We now explain the behaviour of the process $(\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \geq 0, i \in I_A^\lambda}$.

- If $t_0 = 0$, then the process starts from a vacant initial configuration. The match falling on $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at time $\mathbf{a}_\lambda t_1 \in (0, \mathbf{a}_\lambda)$ creates a barrier, see Lemma 9.2, because $t_1 \in (0, 1)$. Then, fires start in $\lfloor \mathbf{n}_\lambda x_k \rfloor$ alternately on the right and on the left of $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at times $\mathbf{a}_\lambda t_k$ for all $k = 2, \dots, K$ and fires spread accross \mathbb{Z} according to the same rules as the (λ, π, A) -FFP.
- If $t_0 > 1$, the process starts from an occupied initial situation. Nothing happens until a match falls in $\lfloor \mathbf{n}_\lambda x_0 \rfloor$ at time $\mathbf{a}_\lambda t_0$ and spreads across I_A^λ (because all the sites are occupied at time $\mathbf{a}_\lambda t_0 -$ and $\lfloor \mathbf{n}_\lambda A \rfloor + 1$ and $-\lfloor \mathbf{n}_\lambda A \rfloor - 1$ are vacants). Next, a match falls on $\lfloor \mathbf{n}_\lambda x_1 \rfloor$ at time $\mathbf{a}_\lambda t_1 \in (\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_0 + 1))$. It then creates a barrier, see Lemma 9.2. Afterwards, matches fall successively in $\lfloor \mathbf{n}_\lambda x_k \rfloor$ at times $\mathbf{a}_\lambda t_k$ for each $k = 2, \dots, K$ and fires spread accross I_A^λ according to the same rules as the (λ, π, A) -FFP.

Consider the event

$$\Omega_{\mathcal{P}}^{S,P}(\lambda, \pi) = \{\forall k \in \{2, \dots, K\}, \exists j \in (x_1)_\lambda, \forall t \in [t_k + \varkappa_{\lambda, \pi}, t_k + 1], \zeta_{\mathbf{a}_\lambda t}^{\lambda, \pi, \mathcal{P}}(j) = 0\}.$$

Lemma 9.3. *Let $\mathcal{P} = (\varepsilon; (x_0, t_0), (x_1, t_1), \dots, (x_K, t_K))$ satisfy (PP) . For each $\lambda \in (0, 1)$ and each $\pi \geq 1$, consider the process $(\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$ defined above.*

If $t_2 - t_1 < t_1 - t_0$, when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$, there holds

$$\lim_{\lambda, \pi} \mathbb{P} \left[\Omega_{\mathcal{P}}^{S,P}(\lambda, \pi) \right] = 1.$$

Proof. Without loss of generality, we assume $x_1 = 0$ and $(x_k)_{k=0,2,\dots,K} \subset [-A, A]$.

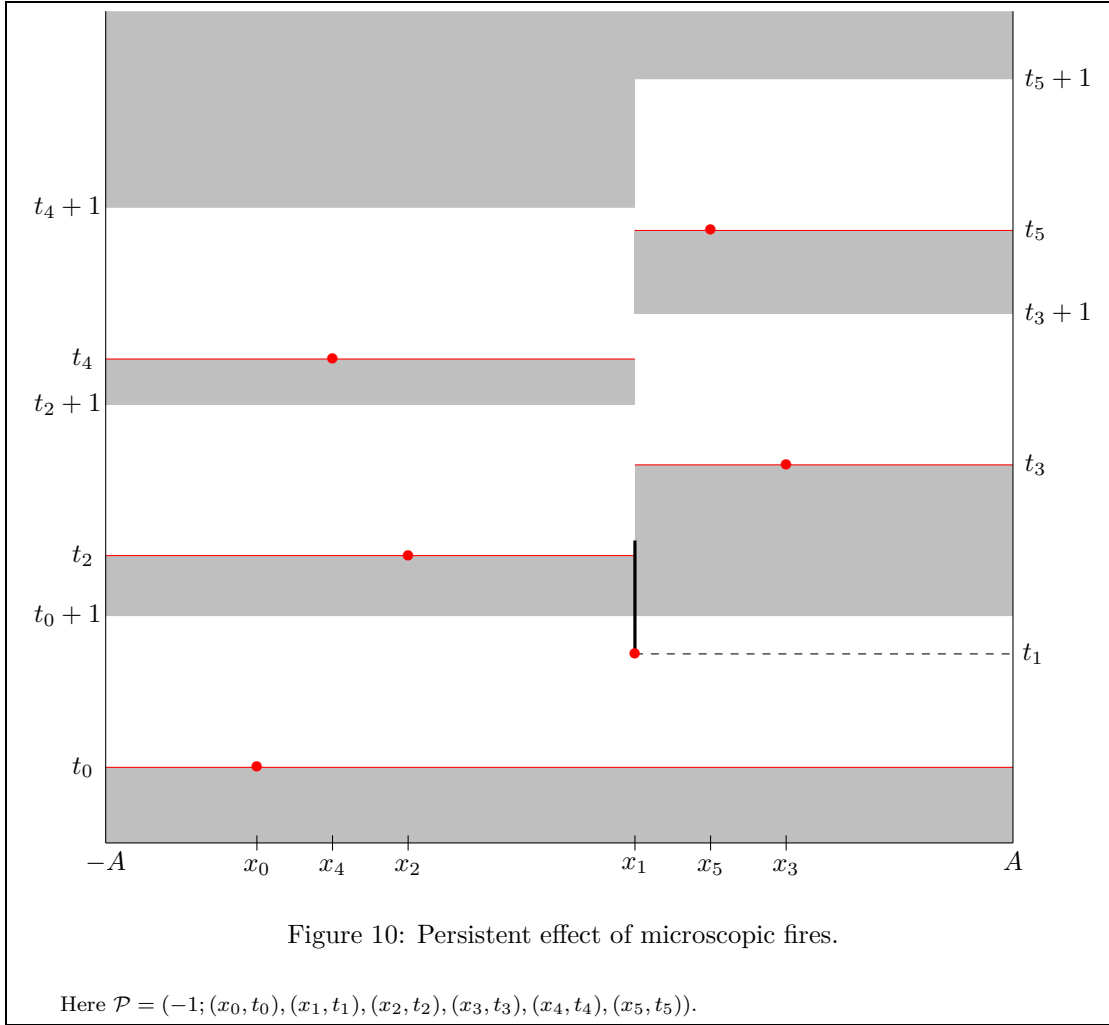
We define, recall Definition 4.7,

$$\Omega_{\lambda,\pi}^{P,A,\mathcal{P}} = \Omega_{\lambda,\pi}^{P,2A,2A}(0, t_1) \cap \bigcap_{k=0,2,\dots,K} \Omega_{\lambda,\pi}^{P,2A,2A}(x_k, t_k).$$

There holds that $\mathbb{P}[\Omega_{\lambda,\pi}^{P,A,\mathcal{P}}]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$ by Lemma 4.3.

In the whole proof, we work on $\Omega_{\lambda,\pi}^{P,A,\mathcal{P}}$ and assume that (λ, π) is sufficiently close to the regime $\mathcal{R}(0)$ in such a way that $\varkappa_{\lambda,\pi} < \min_{i \neq j} |t_i - t_j|$ and $\min_{k=0,2,\dots,K} \lfloor \mathbf{n}_\lambda x_k \rfloor \geq \mathbf{m}_\lambda$.

For simplicity, we assume that $\varepsilon = -1$, $t_0 = 0$ and that K is even. The other cases are treated similarly (see for example Lemma 9.2). Fix $\alpha = 1/K$. We define $\mathcal{M} := ((0, 0), (0, t_1))$, recall Lemma 9.2.



Since $\lfloor \mathbf{n}_\lambda A \rfloor + 1$ and $-\lfloor \mathbf{n}_\lambda A \rfloor - 1$ remain vacant all the time, on $\Omega_{\lambda,\pi}^{P,A,\mathcal{P}}$, a burning tree at time $\mathbf{a}_\lambda t$ is either a front of a fire or has vacant neighbors. Thus, there is no burning tree outside $\cup_{k=1,\dots,K} [\mathbf{a}_\lambda t_k, \mathbf{a}_\lambda(t_k + \varkappa_{\lambda,\pi})]$.

First fire. We put $C^P := C^P((\zeta_t^{\lambda,\pi,\mathcal{P}}(i))_{t \geq 0, i \in \mathbb{Z}}, (0, t_1))$, the destroyed cluster, recall (4.12). Since $t_1 + \varkappa_{\lambda,\pi} < 1$, $C^P \subset \llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ with probability tending to 1 (use Lemma 9.1-1, space/time stationarity and **Micro**(0) in Subsection 4.4). Thus the match falling at time $\mathbf{a}_\lambda t_1$ destroys nothing outside $\llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ and there is no more burning tree in I_A^λ at time $\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda,\pi})$.

Second fire. Since $t_2 > 1$, at least one seed has fallen, during $[0, \mathbf{a}_\lambda t_2)$, on each site of $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ with probability tending to 1 (use Lemma 9.1-4 and space/time stationarity). Since this zone

has not been affected by a fire during the time interval $[0, \mathbf{a}_\lambda t_2)$, this zone is completely occupied at time $\mathbf{a}_\lambda t_2 -$.

Besides, with probability tending to 1, there is (at least) an empty site in $C^P \subset \llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during the time interval $(\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_2 + \varkappa_{\lambda, \pi}))$ because $t_1 + \varkappa_{\lambda, \pi} < t_2 < t_2 + \varkappa_{\lambda, \pi} < t_1 + \Theta_{\mathcal{M}}^{\lambda, \pi}$ with probability tending to 1 (by Lemma 9.2, $\Theta_{\mathcal{M}}^{\lambda, \pi} \simeq t_1 - t_0 = t_1$ and $t_2 - t_1 < t_1 - t_0 = t_1$ by assumption) and because by definition of $\Theta_{\mathcal{M}}^{\lambda, \pi}$, there is an empty site in $C^P \subset \llbracket -\lfloor \alpha \mathbf{m}_\lambda \rfloor, \lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during $[\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_1 + \Theta_{\mathcal{M}}^{\lambda, \pi})]$.

Thus, the fire ignited on $\lfloor \mathbf{n}_\lambda x_2 \rfloor \in \llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda t_2$ burns each site of the zone $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ before $\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda, \pi})$ and does not affect the zone $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$, thanks to $\Omega_{\lambda, \pi}^{P, 2A, 2A}(x_2, t_2)$, as seen in **Macro**(0) in Subsection 4.4.

Third fire. All the sites of $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ are occupied at time $\mathbf{a}_\lambda t_3 -$ with probability tending to 1 (because on $\Omega_{\lambda, \pi}^{P, 2A, 2A}(0, t_1) \cap \Omega_{\lambda, \pi}^{P, 2A, 2A}(x_2, t_2)$, they have not been affected by a fire during $[0, \mathbf{a}_\lambda t_3)$ and because $t_3 > t_2 > 1$, see Lemma 9.1-4).

Next, since $t_3 - t_2 < 1$, the probability that there is a site in $\llbracket -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ where no seed falls during $[\mathbf{a}_\lambda t_2, \mathbf{a}_\lambda(t_2 + 1))$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$ (use Lemma 9.1-1 and space/time stationarity). Thus, with probability tending to 1, there exists a vacant site in $\llbracket -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during $[\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_2 + 1)) \supset [\mathbf{a}_\lambda t_3, \mathbf{a}_\lambda(t_3 + \varkappa_{\lambda, \pi})]$ (because all the sites of $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor \alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ have been made vacant by the fire 2).

Thus, the fire ignited on $\lfloor \mathbf{n}_\lambda x_3 \rfloor \in \llbracket \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ at time $\mathbf{a}_\lambda t_3$ burns each site of $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ before $\mathbf{a}_\lambda(t_3 + \varkappa_{\lambda, \pi})$ and does not affect the zone $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor \rrbracket$ with probability tending to 1, thanks to $\Omega_{\lambda, \pi}^{P, 2A, 2A}(x_3, t_3)$, as seen in **Macro**(0) in Subsection 4.4.

Fourth fire. All the sites of $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ are occupied at time $\mathbf{a}_\lambda t_4 -$ with probability tending to 1 (because on $\Omega_{\lambda, \pi}^{P, 2A, 2A}(0, t_1) \cap \Omega_{\lambda, \pi}^{P, 2A, 2A}(x_2, t_2) \cap \Omega_{\lambda, \pi}^{P, 2A, 2A}(x_3, t_3)$, they have not been affected by a fire during $(\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda t_4)$ and because $t_4 - t_2 - \varkappa_{\lambda, \pi} > 1$, see Lemma 9.1-4 and space/time stationarity).

Since $t_4 - t_3 < 1$, the probability that there is a site in $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor 2\alpha \mathbf{m}_\lambda \rfloor \rrbracket$ where no seed falls during $[\mathbf{a}_\lambda t_3, \mathbf{a}_\lambda(t_3 + 1))$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$ (use Lemma 9.1-1 and space/time stationarity). Hence there is at least one vacant site in $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor 2\alpha \mathbf{m}_\lambda \rfloor \rrbracket$ during $[\mathbf{a}_\lambda(t_3 + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_3 + 1)) \supset [\mathbf{a}_\lambda t_4, \mathbf{a}_\lambda(t_4 + \varkappa_{\lambda, \pi})]$, with probability tending to 1 (because all the sites of $\llbracket \lfloor \alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ have been made vacant by the fire 3).

Thus, the fire ignited on $\lfloor \mathbf{n}_\lambda x_4 \rfloor \in \llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda t_4$ burns each site of the zone $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor 2\alpha \mathbf{m}_\lambda \rfloor - 1 \rrbracket$ before $\mathbf{a}_\lambda(t_4 + \varkappa_{\lambda, \pi})$ and does not affect the zone $\llbracket \lfloor 2\alpha \mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$ with probability tending to 1, thanks to $\Omega_{\lambda, \pi}^{P, 2A, 2A}(x_4, t_4)$, as seen in **Macro**(0) in Subsection 4.4.

Last fire and conclusion. Iterating the procedure, we see that with probability tending to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$, the zone $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor (K\alpha/2)\mathbf{m}_\lambda \rfloor - 1 \rrbracket = \llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor \mathbf{m}_\lambda/2 \rfloor - 1 \rrbracket$ is completely occupied at time $\mathbf{a}_\lambda t_K -$ and there is at least one vacant site in $\llbracket \lfloor (K-1)\alpha/2\mathbf{m}_\lambda \rfloor + 1, \lfloor \mathbf{m}_\lambda/2 \rfloor \rrbracket$ during the time interval $[\mathbf{a}_\lambda(t_{K-1} + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_{K-1} + 1)) \supset [\mathbf{a}_\lambda t_K, \mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi})]$. Thus, the fire ignited on $\lfloor \mathbf{n}_\lambda x_K \rfloor \in \llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda t_K$ destroys each site of the zone $\llbracket -\lfloor \mathbf{n}_\lambda A \rfloor, -\lfloor \mathbf{m}_\lambda/2 \rfloor - 1 \rrbracket$ before $\mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi})$ and does not affect the zone $\llbracket \lfloor \mathbf{m}_\lambda/2 \rfloor, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$.

Finally, the probability that there is at least one site in $\llbracket -\mathbf{m}_\lambda, -\mathbf{m}_\lambda/2 \rrbracket$ with no seed falling during $[\mathbf{a}_\lambda t_K, \mathbf{a}_\lambda(t_K + 1))$ tends to 1 (by Lemma 9.1-1). Consequently, the probability that there is a vacant site in $\llbracket -\mathbf{m}_\lambda, -\mathbf{m}_\lambda/2 \rrbracket$ during $[\mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_K + 1))$ tends to 1. All this implies the claim. \square

9.4 Heart of the proof

9.4.1 The coupling

We are going to construct a coupling between the (λ, π, A) -FFP (on the time interval $[0, \mathbf{a}_\lambda T]$) and the A -LFFP(0) (on $[0, T]$). Let π_M be a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $dx dt$.

First, we take for the matches of the discrete process the Poisson processes

$$N_t^M(i) = \pi_M([i/\mathbf{n}_\lambda, (i+1)/\mathbf{n}_\lambda) \times [0, t/\mathbf{a}_\lambda])$$

for all $i \in \mathbb{Z}$ and $t \in [0, T]$.

We call $n := \pi_M([0, T] \times [-A, A])$ and we consider the marks $(T_q, X_q)_{q=1, \dots, n}$ of π_M ordered in such a way that $0 < T_1 < \dots < T_n < T$.

Next, we introduce two families of i.i.d. Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ with respective parameters 1 and π , independent of π_M .

The (λ, π, A) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in I_A^\lambda}$ is built from the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$, the match processes $(N_t^M(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$.

Finally, we build the A -LFFP(0) $(Z_t(x), H_t(x), F_t(x))_{t \in [0, T], x \in [-A, A]}$ from π_M and observe that it is independent of $(N_t^S(i))_{t \in [0, a_\lambda T], i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \in [0, a_\lambda T], i \in \mathbb{Z}}$.

Observe that if a match falls on some X_q at time T_q for the A -LFFP(0), it also falls on $\lfloor \mathbf{n}_\lambda X_q \rfloor$ at time $a_\lambda T_q$ in the discrete process.

9.4.2 A favorable event

We set $T_0 = 0$ and introduce

$$\mathcal{T}_M = \{T_0, T_1, \dots, T_n\} \text{ and } \mathcal{B}_M = \{X_1, \dots, X_n\}$$

as well as the set \mathcal{C}_M of connected components of $[-A, A] \setminus \mathcal{B}_M$ (sometimes referred to as cells). We also introduce

$$\mathcal{S}_M = \{2t - s : s, t \in \mathcal{T}_M, s < t\}$$

which has to be seen as the set of the possible extinction times of the microscopic fires, recall Lemma 9.2.

For $\alpha > 0$, we consider the event

$$\Omega_M(\alpha) = \left\{ \min_{\substack{s, t \in \mathcal{T}_M \cup \mathcal{S}_M \\ s \neq t}} |t - s| \geq 2\alpha, \min_{s, t \in \mathcal{T}_M \cup \mathcal{S}_M} |t - (s + 1)| \geq 2\alpha, \min_{\substack{x, y \in \mathcal{B}_M \cup \{-A, A\} \\ x \neq y}} |x - y| \geq 2\alpha \right\}$$

which clearly satisfies $\lim_{\alpha \rightarrow 0} \mathbb{P}[\Omega_M(\alpha)] = 1$. For any given $\alpha > 0$, there exists $\lambda_\alpha > 0$ such that for all $\lambda \in (0, \lambda_\alpha)$, on $\Omega_M(\alpha)$, there holds that

- for all $x, y \in \mathcal{B}_M \cup \{-A, A\}$, with $x \neq y$, $(x)_\lambda \cap (y)_\lambda = \emptyset$;
- the family $\{c_\lambda, c \in \mathcal{C}_M\} \cup \{(x)_\lambda, x \in \mathcal{B}_M\}$ is a partition of I_A^λ .

For $q \in \{1, \dots, n\}$, using the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, we build, recall Definition 4.6, $(\zeta_t^{\lambda, \pi, q}(i))_{t \geq 0, i \in \mathbb{Z}}$ (the propagation process ignited at (X_q, T_q)), $(i_t^{q, +})_{t \geq 0}$ and $(i_t^{q, -})_{t \geq 0}$ (the corresponding right and left fronts) and $(T_i^q)_{i \in \mathbb{Z}}$ (the associated burning times). We also use $\Omega_{\lambda, \pi}^{P, 2A, 2A}(X_q, T_q)$, recall Definition 4.7. We set

$$\Omega_A^{S, P}(\lambda, \pi) = \bigcap_{q=1, \dots, n} \Omega_{\lambda, \pi}^{P, 2A, 2A}(X_q, T_q).$$

Since π_M is independent of the processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$, Lemma 4.3 implies that $\mathbb{P}[\Omega_A^{S, P}(\lambda, \pi)]$ tends to 1 when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

Let $q \in \{1, \dots, n\}$. We call \mathcal{U}_q the set of all possible $\mathcal{P} = (\varepsilon; (x_0, t_0), (X_q, T_q), \dots, (x_K, t_K))$ satisfying (PP) where $\{t_0, t_2, \dots, t_K\} \subset \mathcal{T}_M$, $\{x_0, x_2, \dots, x_K\} \subset \mathcal{B}_M$ with $T_q - t_0 > t_2 - T_q$ and with $\varepsilon \in \{-1, 1\}$. For $\mathcal{P} \in \mathcal{U}_q$, we introduce the event $\Omega_{\mathcal{P}}^{S, P}(\lambda, \pi)$, defined as in Subsection 9.3, with the Poisson processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$. Then we put

$$\Omega_1^{S, P}(\lambda, \pi) = \bigcap_{q=1}^n \bigcap_{\mathcal{P} \in \mathcal{U}_q} \Omega_{\mathcal{P}}^{S, P}(\lambda, \pi),$$

which satisfies $\lim_{\lambda, \pi} \mathbb{P}[\Omega_1^{S, P}(\lambda, \pi)] = 1$ when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(p)$, thanks to Lemma 9.3.

We also consider the event $\Omega_2^S(\lambda, \pi)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M$ with $0 < t_2 - t_1 < 1$, for all $q = 1, \dots, n$, there are

$$-\mathbf{m}_\lambda < i_1 < 0 < i_2 < \mathbf{m}_\lambda$$

such that $N_{\mathbf{a}_\lambda(t_2 + \varkappa_{\lambda, \pi})}^S(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_j) - N_{\mathbf{a}_\lambda t_1}^S(\lfloor \mathbf{n}_\lambda X_q \rfloor + i_j) = 0$ for $j = 1, 2$. There holds that $\mathbb{P}[\Omega_2^S(\lambda, \pi)]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$. Indeed, it suffices to prove that almost surely, $\lim_{\lambda \rightarrow 0, \pi \rightarrow \infty} \mathbb{P}[\Omega_2^S(\lambda, \pi) \mid \pi_M] = 1$. Since there are a.s. finitely many possibilities for q, t_1, t_2 and since π_M is independent of $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$, it suffices to work with a fixed $q \in \{1, \dots, n\}$ and some fixed $0 < t_2 - t_1 < 1$. The result then follows from Lemma 9.1-1 together with space/time stationarity.

Next we introduce the event $\Omega_3^S(\lambda, \pi)$ on which the following conditions hold: for all $t_1, t_2 \in \mathcal{T}_M \cup \mathcal{S}_M$,

- if $t_2 - t_1 > 1$, for all $c \in \mathcal{C}_M$, for all $i \in c_\lambda$ with $N_{\mathbf{a}_\lambda t_2}^S(i) - N_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})}^S(i) > 0$;
- if $t_2 - t_1 > 1$, for all $x \in \mathcal{B}_M$, for all $i \in (x)_\lambda$ with $N_{\mathbf{a}_\lambda t_2}^S(i) - N_{\mathbf{a}_\lambda(t_1 + \varkappa_{\lambda, \pi})}^S(i) > 0$.

There holds that $\mathbb{P}[\Omega_3^S(\lambda, \pi)]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$. As previously, it suffices to work with some fixed $t_1, t_2 \in \mathcal{T}_M$, $x \in \mathcal{B}_M$ and $c = (a, b) \subset (-A, A)$. Observing that $|c_\lambda| \simeq (b - a)\mathbf{n}_\lambda$ and that $|(x)_\lambda| \simeq 2\mathbf{m}_\lambda$, Lemma 9.1 and space/time stationarity shows the result.

We also need $\Omega_4^{S,P}(\gamma, \lambda, \pi)$, defined for $\gamma > 0$ as follows: for all $q = 1, \dots, n$, for all $\mathcal{M} = ((x_0, t_0), (X_q, T_q))$ such that $t_0 \in \mathcal{T}_M$ with $t_0 < T_q < t_0 + 1$ and $x_0 \in \mathcal{B}_M \setminus \{X_q\}$, there holds that $|\Theta_{\mathcal{M}}^{\lambda, \pi} - (T_q - t_0)| < \gamma$. Here, $\Theta_{\mathcal{M}}^{\lambda, \pi}$ is defined as in Lemma 9.2 with the seed processes family $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes family $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$. Lemma 9.2 directly implies that for any $\gamma > 0$, $\mathbb{P}[\Omega_4^{S,P}(\gamma, \lambda, \pi)]$ tends to 1 as $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$.

We finally introduce the event

$$\Omega(\alpha, \gamma, \lambda, \pi) = \Omega_M(\alpha) \cap \Omega_A^{S,P}(\lambda, \pi) \cap \Omega_1^{S,P}(\lambda, \pi) \cap \Omega_2^S(\lambda, \pi) \cap \Omega_3^S(\lambda, \pi) \cap \Omega_4^{S,P}(\gamma, \lambda, \pi).$$

We have shown that for any $\delta > 0$, there exists $\alpha \in (0, 1)$ such that for any $\gamma > 0$, there holds that $\mathbb{P}[\Omega(\alpha, \gamma, \lambda, \pi)] > 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

9.4.3 Heart of the proof

We now handle the main part of the proof.

Consider the A -LFFP(0). Observe that by construction, we have, for $c \in \mathcal{C}_M$ and $x, y \in c$, $Z_t(x) = Z_t(y)$ for all $t \in [0, T]$, thus we can introduce $Z_t(c)$.

If $x \in \mathcal{B}_M$, it is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_M$ and then we set $Z_t(x_-) = Z_t(c_-)$ and $Z_t(x_+) = Z_t(c_+)$ for all $t \in [0, T]$.

If $x \in (-A, A) \setminus \mathcal{B}_M$, we put $Z_t(x_-) = Z_t(x_+) = Z_t(x)$ for all $t \in [0, T]$.

For $x \in \mathcal{B}_M$ and $t \geq 0$ we set $\tilde{H}(x) = \min(H_t(x), 1 - Z_t(x), 1 - Z_t(x_-), 1 - Z_t(x_+))$.

Actually $Z_t(x)$ always equals either $Z_t(x_-)$ or $Z_t(x_+)$ and these can be distinct only at a point where has occurred a microscopic fire (that is if $x = X_q$ for some $q \in \{1, \dots, n\}$ with $T_q < t$ and $Z_{T_q-}(X_q) < 1$).

For all $x \in (-A, A)$ and $t \in [0, T]$, we put

$$\tau_t(x) = \sup \{s \leq t : Z_s(x_+) = Z_s(x_-) = Z_s(x) = 0\} \in \mathcal{T}_M.$$

For $c \in \mathcal{C}_M$ and $t \in [0, T]$, we can define $\tau_t(c)$ as usual with the convention $Z_{0-}(x) = 1$ for all $x \in [-A, A]$.

Observe that

$$\text{for } x \notin \mathcal{B}_M, Z_t(x) = \min(t - \tau_t(x), 1) \text{ for all } t \in [0, T], \quad (9.1)$$

$$\text{for } q = 1, \dots, n, Z_t(X_q) = \min(t - \tau_t(X_q), 1) \text{ for all } t \in [0, T_q]. \quad (9.2)$$

We also define, for all $t \in [0, T]$, all $i \in I_\lambda^A$,

$$\rho_t^{\lambda, \pi}(i) = \sup \left\{ s \leq t : \eta_{\mathbf{a}_\lambda s-}^{\lambda, \pi}(i) = 2 \right\}$$

with the convention $\eta_{0-}^{\lambda, \pi}(i) = 2$ and $\eta_0^{\lambda, \pi}(i) = 0$.

For $t \in [0, T]$, consider the event

$$\Omega_t^{\lambda, \pi} = \left\{ \forall s \in [0, t] \setminus \bigcup_{q=1}^n [T_q, T_q + \varkappa_{\lambda, \pi}), \forall c \in \mathcal{C}_M, \forall i \in c_\lambda, |\rho_s^{\lambda, \pi}(i) - \tau_s(c)| \leq \varkappa_{\lambda, \pi} \right\}.$$

Lemma 9.4. *Let $\alpha > \gamma > 0$. For all $\lambda \in (0, \lambda_\alpha)$ and $\pi \geq 1$ such that $\varkappa_{\lambda, \pi} \leq \alpha$, $\Omega_T^{\lambda, \pi}$ a.s. holds on $\Omega(\alpha, \gamma, \lambda, \pi)$.*

Proof. We work on $\Omega(\alpha, \gamma, \lambda, \pi)$ and assume that $\lambda \in (0, \lambda_\alpha)$ and $\pi \geq 1$ are such that $\varkappa_{\lambda, \pi} \leq \alpha$. Clearly, $\tau_0(c) = 0$ and $\rho_0^{\lambda, \pi}(i) = 0$ for all $c \in \mathcal{C}_M$ and all $i \in I_A^\lambda$, so that $\Omega_0^{\lambda, \pi}$ a.s. holds. We will show that for $q = 0, \dots, n-1$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$. The extension to $\Omega_T^{\lambda, \pi}$ will be straightforward and will be omitted.

We thus fix $q \in \{0, \dots, n-1\}$ and assume $\Omega_{T_q}^{\lambda, \pi}$. We repeatedly use below that for all $k \leq q$, on the time interval (T_k, T_{k+1}) , there are no fires at all (in $[-A, A]$) for the A -LFFP(0) and, on $\Omega_A^{S, P}(\lambda, \pi)$, no burning tree at all (in I_A^λ) during $(\mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda T_{k+1})$ for the (λ, π, A) -FFP.

Besides, $\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) = \eta_{\mathbf{a}_\lambda T_q}^{\lambda, \pi}(i)$ for all $i \in I_A^\lambda \setminus \{\lfloor \mathbf{n}_\lambda X_q \rfloor\}$ while

$$\eta_{\mathbf{a}_\lambda T_q}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 2\mathbf{1}_{\{\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda X_q \rfloor) = 1\}}.$$

Step 1. Here we prove that, on $\Omega_{T_q}^{\lambda, \pi}$, for all $1 \leq k < q$, if $D_{T_k-}(X_k) = [a, b]$, for some $a < b$, $a, b \in \mathcal{B}_M \cup \{-A, A\}$, then

$$\eta_{\mathbf{a}_\lambda T_k + T_{i-}^k - \lfloor \mathbf{n}_\lambda X_k \rfloor}^{\lambda, \pi}(i) = 2$$

for all $i \in [a, b]_\lambda$.

On the one hand, by construction, for all $c \in \mathcal{C}_M$, $c \subset (a, b)$, we have $\tau_{T_k}(c) = T_k$. By $\Omega_{T_q}^{\lambda, \pi} \subset \Omega_{T_k + \varkappa_{\lambda, \pi}}^{\lambda, \pi}$, we deduce that $T_k \leq \rho_{T_k + \varkappa_{\lambda, \pi}}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1) \leq T_k + \varkappa_{\lambda, \pi}$.

On the other hand, recall Lemma 4.3: on $\Omega_{\lambda, \pi}^{P, 2A, 2A}(X_k, T_k)$, a burning tree is either a front or has vacant neighbors. Recall that there is no burning tree at all in I_A^λ at time $\mathbf{a}_\lambda T_k-$. Assume for example that there is a site $i \in \llbracket \lfloor \mathbf{n}_\lambda X_k \rfloor, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1 \rrbracket$ such that $\eta_{\mathbf{a}_\lambda T_k + T_{i-}^k - \lfloor \mathbf{n}_\lambda X_k \rfloor}^{\lambda, \pi}(i) = 0$. Then the fire starting at $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$ does not affect that zone $\llbracket i, \lfloor \mathbf{n}_\lambda A \rfloor \rrbracket$, as seen in **Macro**(0) in Subsection 4.4. This especially implies that $\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1) \leq 1$ for all $t \in [T_k, T_k + \varkappa_{\lambda, \pi}]$ (because no other match falls on I_A^λ during $[\mathbf{a}_\lambda T_k, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi})]$) whence $\rho_{T_k + \varkappa_{\lambda, \pi}}^{\lambda, \pi}(\lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda - 1) < T_k$, a contradiction.

Step 2. We show that on $\Omega_{T_q}^{\lambda, \pi}$, for all $c \in \mathcal{C}_M$, all $i \in c_\lambda$,

$$\underline{\eta}_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) \leq \eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) \leq \overline{\eta}_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) \quad (9.3)$$

where

$$\begin{aligned} \underline{\eta}_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) &= \min(N_{\mathbf{a}_\lambda T_q-}^S(i) - N_{\mathbf{a}_\lambda \tau_{T_q-}(c) + \varkappa_{\lambda, \pi}}^S(i), 1), \\ \overline{\eta}_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) &= \min(N_{\mathbf{a}_\lambda T_q-}^S(i) - N_{\mathbf{a}_\lambda \tau_{T_q-}(c)}^S(i), 1). \end{aligned}$$

Indeed, thanks to $\Omega_A^{S, P}(\lambda, \pi) \cap \Omega_M(\alpha)$, there is no burning tree in I_A^λ at time $\mathbf{a}_\lambda T_q-$. Furthermore, for $c \in \mathcal{C}_M$, by $\Omega_{T_q}^{\lambda, \pi}$, we have

$$\tau_{T_q-}(c) \leq \rho_{T_q-}^{\lambda, \pi}(i) \leq \tau_{T_q-}(c) + \varkappa_{\lambda, \pi} \text{ for all } i \in c_\lambda.$$

By definition, no fire can affect the site i during $(\mathbf{a}_\lambda \rho_{T_q-}^{\lambda, \pi}(i), \mathbf{a}_\lambda T_q)$ whence (9.3).

Step 3. We show here that if $Z_{T_q-}(X_q) < 1$, there exist $j_1, j_2 \in (X_q)_\lambda$ such that

$$j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$$

$$\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(j_1) = \eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}(j_2) = 0 \text{ for all } t \in [T_q, T_q + \varkappa_{\lambda, \pi}].$$

Indeed, since no match falls on X_q during the time interval $[0, T_q)$, we have $\tau_{T_q-}(X_q) = T_q - Z_{T_q-}(X_q) = T_k$, for some $0 \leq k < q$. Observe that $Z_{T_q-}(X_q) < 1$ implies that $T_q - \tau_{T_q-}(X_q) < 1$.

- If $1 \leq k < q$, then, by construction, we have $X_q \in \overset{\circ}{D}_{T_k-}(X_k) = (a, b)$, for some $a, b \in \mathcal{B}_M \cup \{-A, A\}$. By $\Omega_M(\alpha)$, we have $|a - X_k| \wedge |b - X_k| > 2\alpha$ whence $(X_q)_\lambda \subset [a, b]_\lambda$. We deduce from Step 1 that $\eta_{\mathbf{a}_\lambda T_k + T_i^k - \lfloor \mathbf{n}_\lambda X_k \rfloor}^{\lambda, \pi}(i) = 2$ for all $i \in (X_q)_\lambda$. Since we work on $\Omega_2^S(\lambda, \pi)$ and $T_k, T_q \in \mathcal{T}_M$, we know that there are some sites

$$\lfloor \mathbf{n}_\lambda X_k \rfloor - \mathbf{m}_\lambda < j_1 < \lfloor \mathbf{n}_\lambda X_k \rfloor < j_2 < \lfloor \mathbf{n}_\lambda X_k \rfloor + \mathbf{m}_\lambda$$

such that no seed has fallen on j_1 and j_2 during $[\mathbf{a}_\lambda \tau_{T_q-}(X_q), \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$. Since they are made vacant by the fire k during the time interval $[\mathbf{a}_\lambda T_k, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi})]$, we deduce that they remain vacant during $[\mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})] \supset [\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$.

- If $k = 0$, that is if $\tau_{T_q-}(X_q) = 0$ we deduce that $T_q < 1$. We conclude using $\Omega_2^S(\lambda, \pi)$ that there are $j_1 < \lfloor \mathbf{n}_\lambda X_q \rfloor < j_2$ with $j_1, j_2 \in (X_q)_\lambda$ where no seed fall during $[0, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$. Since all the sites are vacant at time 0, we deduce that j_1 and j_2 remain vacant until $\mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})$.

Step 4. Next we check that if $Z_{T_q-}(c) = 1$ for some $c \in \mathcal{C}_M$, then

$$\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) = 1 \text{ for all } i \in c_\lambda.$$

Recalling (9.1), we see that $Z_{T_q-}(c) = 1$ implies that $T_q - \tau_{T_q-}(c) \geq 1$ and $T_q - \tau_{T_q-}(c) \geq 1 + 2\alpha$ by $\Omega_M(\alpha)$. Using Step 2, we see that for all $i \in c_\lambda$,

$$\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) \geq \underline{\eta}_{\mathbf{a}_\lambda T_q}^{\lambda, \pi}(i) = \min(N_{\mathbf{a}_\lambda T_q-}^S(i) - N_{\mathbf{a}_\lambda \tau_{T_q-}(c) + \varkappa_{\lambda, \pi}}^S(i), 1).$$

We conclude using $\Omega_3^S(\lambda, \pi)$ that for all $i \in c_\lambda$, $\underline{\eta}_{\mathbf{a}_\lambda T_q}^{\lambda, \pi}(i) = 1$ whence $\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) = 1$, as desired.

Step 5. We now prove that if $\tilde{H}_{T_q-}(x) = 0$ for some $x \in \mathcal{B}_M$, then

$$\eta_{\mathbf{a}_\lambda T_q-}^{\lambda, \pi}(i) = 1 \text{ for all } i \in (x)_\lambda.$$

Preliminary considerations. Let $k \in \{1, \dots, n\}$ such that $x = X_k$, which is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_M$. We know that $\tilde{H}_{T_q-}(x) = 0$, whence $H_{T_q-}(x) = 0$ and $Z_{T_q-}(x) = Z_{T_q-}(c_+) = Z_{T_q-}(c_-) = 1$. This implies that $T_q \geq 1$ (because $Z_t(x) = t$ for all $t < 1$ and all $x \in [-A, A]$) and thus $T_q \geq 1 + 2\alpha$ due to $\Omega_M(\alpha)$.

No fire has concerned $j_g = \lfloor \mathbf{n}_\lambda X_k \rfloor - \mathbf{m}_\lambda - 1 \in (c_-)_\lambda$ during $(\mathbf{a}_\lambda \rho_{T_q-}^{\lambda, \pi}(j_g), \mathbf{a}_\lambda T_q)$. By $\Omega_{T_q}^{\lambda, \pi}$, we deduce that $\tau_{T_q-}(c_-) \leq \rho_{T_q-}^{\lambda, \pi}(j_g) \leq \tau_{T_q-}(c_-) + \varkappa_{\lambda, \pi}$. Recalling (9.1), $Z_{T_q-}(c_-) = 1$ implies that $\tau_{T_q-}(c_-) \leq T_q - 1$ whence, by $\Omega_M(\alpha)$, there holds that $\tau_{T_q-}(c_-) < T_q - 1 - 2\alpha$. Using a similar argument for $j_d = \lfloor \mathbf{n}_\lambda X_k \rfloor + \mathbf{m}_\lambda + 1 \in (c_+)_\lambda$, we conclude that no match falling outside $(X_k)_\lambda$ can affect $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_q - 1 - \alpha), \mathbf{a}_\lambda T_q)$ (because to affect $(X_k)_\lambda$, a match falling outside $(X_k)_\lambda$ needs to cross j_d or j_g).

Case 1. First assume that $k \geq q$. Then we know that no fire has fallen on $(X_k)_\lambda$ during $[0, \mathbf{a}_\lambda T_q)$. Due to the preliminary considerations, we deduce that no fire at all has concerned $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_q - 1 - \alpha), \mathbf{a}_\lambda T_q)$. Using $\Omega_3^S(\lambda, \pi)$, we conclude that $(X_k)_\lambda$ is completely occupied at time $\mathbf{a}_\lambda T_q-$.

Case 2. Assume that $k < q$ and $Z_{T_k-}(X_k) = 1$, so that there already has been a macroscopic fire in $(X_k)_\lambda$ (at time $\mathbf{a}_\lambda T_k$). Since $Z_{T_k}(X_k) = 0$ and $Z_{T_q-}(X_k) = 1$, we deduce that $T_q - T_k \geq 1$,

whence $T_q - T_k \geq 1 + 2\alpha$ as usual. Since there is no more burning tree in $(X_k)_\lambda$ at time $\mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})$, thanks to $\Omega_{\lambda,\pi}^{P,A}(X_k, T_k)$, we conclude as in Case 1 that no fire at all has concerned $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_q - 1 - \alpha), \mathbf{a}_\lambda T_q)$, which implies the claim by $\Omega_3^S(\lambda, \pi)$.

Case 3. Assume that $k < q$ and $Z_{T_k-}(X_k) < 1$ and $T_q - T_k \geq 1$, whence $T_q - T_k \geq 1 + 2\alpha$ due to $\Omega_M(\alpha)$. Then there already has been a microscopic fire in $(X_k)_\lambda$ (at time $\mathbf{a}_\lambda T_k$). But there are no fire in $(X_k)_\lambda$ during $(\mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi}), \mathbf{a}_\lambda T_q) \supset (\mathbf{a}_\lambda(T_q - 1 - \alpha), \mathbf{a}_\lambda T_q)$ and we conclude as in Case 2.

Case 4. Assume finally that $k < q$ and $Z_{T_k-}(X_k) < 1$ and $T_q - T_k < 1$, whence $T_q - T_k < 1 - 2\alpha$ due to $\Omega_M(\alpha)$. There has been a microscopic fire in $(X_k)_\lambda$ (at time $\mathbf{a}_\lambda T_k$). Since $H_{T_q-}(X_k) = 0$, we deduce that $T_k + Z_{T_k}(X_k) \leq T_q$, whence $T_k + Z_{T_k}(X_k) \leq T_q - 2\alpha$ by $\Omega_M(\alpha)$. There is $l < k$ such that $\tau_{T_k-}(X_k) = T_l$. We set $\mathcal{M} := ((X_l, T_l), (X_k, T_k))$, recall Subsection 9.2 (if $l = 0$ i.e. $\tau_{T_k-}(X_k) = 0$, set for example $X_0 = 0$).

We first show that

$$(\eta_t^{\lambda,\pi}(i))_{t \in [\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})], i \in (X_k)_\lambda} = (\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \in [\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})], i \in (X_k)_\lambda}. \quad (9.4)$$

Here, the process $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \in [\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda,\pi})], i \in (X_k)_\lambda}$ is built as in Subsection 9.2 using the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$.

- We first assume that $T_l \geq 1$, whence $T_l \geq 1 + 2\alpha$ by $\Omega_M(\alpha)$. Since no match has fallen on $(X_k)_\lambda$ during $[0, \mathbf{a}_\lambda T_l]$ and since $Z_{T_l-}(X_k) = 1$, the zone $(X_k)_\lambda$ is completely occupied at time $\mathbf{a}_\lambda T_l-$, recall Case 1. Thus, $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}$ are equal on $(X_k)_\lambda$ at time $\mathbf{a}_\lambda T_l$. By Step 1, we deduce, that

$$\eta_{\mathbf{a}_\lambda T_l + T_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda,\pi}(i) = 2 \text{ for all } i \in D_{T_l-}(X_l)_\lambda.$$

Since $(X_k)_\lambda \subset D_{T_l-}(X_l)_\lambda$, we deduce that $\eta_{\mathbf{a}_\lambda T_l + T_{i - \lfloor \mathbf{n}_\lambda X_l \rfloor}}^{\lambda,\pi}(i) = 2$ for all $i \in (X_k)_\lambda$. Observe that, with our coupling, the fire l propagates according to the same processes in both cases. Since seeds fall on $(X_k)_\lambda$ according to the same processes and since $(\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}$ evolve according to the same rules, we deduce that they remain equals on $(X_k)_\lambda$ during $[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_l + \varkappa_{\lambda,\pi})]$. Next, no fire affects the zone $(X_k)_\lambda$ during $[\mathbf{a}_\lambda(T_l + \varkappa_{\lambda,\pi}), \mathbf{a}_\lambda T_k]$ (because to affect the zone $(X_k)_\lambda$, we need $Z_{s-}(c_-) = 1$ or $Z_{s-}(c_+) = 1$ for some $s \in (T_l, T_k)$ whereas $Z_s(c_-) = Z_s(c_+) = s - T_l$ for all $s \in [T_l, T_k]$) and since seeds fall on $(X_k)_\lambda$ according to the same processes, they are again equal during this time interval. Finally, $C^P((\zeta_t^{\lambda,\pi,\mathcal{M}}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_k, T_k)) \subset (X_k)_\lambda$, recall Lemma 9.2. We deduce (9.4) because the match falling on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$ destroys the same zone, since the two processes evolve with the same rules on $(X_k)_\lambda$.

- If $T_l < 1$, then by construction $l = 0$ and $\tau_{T_k-}(X_k) = 0$. We also deduce (9.4) using similar arguments as above (this case is easier).

Consider now the zone $C^P = C^P((\eta_t^{\lambda,\pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_k, T_k))$ destroyed by the match falling on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k$. This zone is completely occupied at time $\mathbf{a}_\lambda(T_k + \Theta_{\mathcal{M}}^{\lambda,\pi})$: this follows from the definition of $\Theta_{\mathcal{M}}^{\lambda,\pi}$, see Lemma 9.2, from (9.4) and from the preliminary considerations. Using $\Omega_4^S(\gamma, \lambda, \pi)$, we deduce that $T_k + \Theta_{\mathcal{M}}^{\lambda,\pi} \leq T_k + Z_{T_k-}(X_k) + \gamma < T_q$, since $\gamma < \alpha$. Hence C^P is completely occupied at time $\mathbf{a}_\lambda T_q-$.

Consider now $i \in (X_k)_\lambda \setminus C^P$. Then i has not been killed by the fire starting at $\lfloor \mathbf{n}_\lambda X_k \rfloor$. Thus i cannot have been killed during $(\mathbf{a}_\lambda(T_q - 1 - \alpha), \mathbf{a}_\lambda T_q)$ (due to the preliminary considerations) and we conclude, using $\Omega_3^S(\lambda, \pi)$, that i is occupied at time $\mathbf{a}_\lambda T_q-$. This implies the claim.

Step 6. Let us now prove that if $\tilde{H}_{T_q-}(x) > 0$ and $Z_{T_q-}(x_+) = 1$ for some $x \in \mathcal{B}_M$, there is $i_1 \in (x)_\lambda$ such that $\eta_{\mathbf{a}_\lambda t}^{\lambda,\pi}(i_1) = 0$ for all $t \in [T_q, T_q + \varkappa_{\lambda,\pi}]$. Recall that x is at the boundary of two cells c_-, c_+ .

We have either $H_{T_q-}(x) > 0$ or $Z_{T_q-}(c_-) < 1$ (because $Z_{T_q-}(c_+) = 1$ by assumption). Clearly, $x = X_k$ for some $k < q$, with $Z_{T_k-}(X_k) < 1$ (else, we would have $H_t(x) = 0$ and $Z_t(c_-) = Z_t(c_+)$ for all $t \in [0, T_q]$). Thus, recalling (9.1), $T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = T_l$, for some $l < k$.

As checked in case 4 in the previous Step, on $\Omega(\alpha, \gamma, \lambda, \pi)$, setting $\mathcal{M} = ((X_l, T_l), (X_k, T_k))$ (if $l = 0$, set for example $X_0 = 0$)

$$(\eta_t^{\lambda, \pi}(i))_{t \in [\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi})], i \in (X_k)_\lambda} = (\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \in [\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi})], i \in (X_k)_\lambda}$$

where the process $(\zeta_t^{\lambda, \pi, \mathcal{M}}(i))_{t \in [\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi})], i \in (X_k)_\lambda}$ is built as in Subsection 9.2 using the seed processes $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and the propagation processes $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$. Hence, either $l = 0$ whence $\eta_0^{\lambda, \pi}(i) = 0$ for all $i \in (X_k)_\lambda$ or all the sites in $(X_k)_\lambda$ burn at least on time during $[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_l + \varkappa_{\lambda, \pi})]$.

Case 1. Assume first that $H_{T_q-}(x) > 0$. Then by construction, there holds $T_k + Z_{T_k-}(X_k) > T_q > T_k$, whence by $\Omega_M(\alpha)$, $T_k + Z_{T_k-}(X_k) > T_q + 2\alpha > T_k + 4\alpha$.

Consider $C^P = C^P((\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}, (X_k, T_k))$ the zone destroyed by the match falling on $[\mathbf{n}_\lambda X_k]$ at time $\mathbf{a}_\lambda T_k$. By $\Omega_2^S(\lambda, \pi)$ and (9.4), we have $C^P \subset (X_k)_\lambda$ (because $T_k - Z_{T_k-}(X_k)$ and T_k belong to \mathcal{T}_M , because $0 < Z_{T_k-}(X_k) < 1$ and because all the sites in $(X_k)_\lambda$ have been made vacant during $[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_l + \varkappa_{\lambda, \pi})]$).

By Definition of $\Theta_{\mathcal{M}}^{\lambda, \pi}$, see Lemma 9.2 and by (9.4), we deduce that C^P is not completely occupied at time $\mathbf{a}_\lambda(T_k + \Theta_{\mathcal{M}}^{\lambda, \pi})$ (because in both cases, seeds fall on $(X_k)_\lambda$ according to the same processes). But by $\Omega_4^{S, P}(\gamma, \lambda, \pi)$, we see that $\Theta_{\mathcal{M}}^{\lambda, \pi} \geq Z_{T_k-}(X_k) - \gamma$, whence $T_k + \Theta_{\mathcal{M}}^{\lambda, \pi} \geq T_k + Z_{T_k-}(X_k) - \gamma + 2\alpha > T_q + \varkappa_{\lambda, \pi}$ since $\gamma < \alpha$ and $\varkappa_{\lambda, \pi} < \alpha$. All this implies that there is a vacant site in C^P during $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$.

Case 2. Assume next that $H_{T_q-}(x) = 0$ and that $T_q - T_l < 1$ (whence $T_q - T_l < 1 - 2\alpha$).

- If $l \geq 1$, recall that a match has fallen (in the limit process) on $X_l \in \mathcal{B}_M$ at time $T_l \in \mathcal{T}_M$ with $X_k \in \mathring{D}_{T_l-}(X_l)$. Since T_l and T_q belong to \mathcal{T}_M and since their difference is smaller than 1 by assumption, $\Omega_2^S(\lambda, \pi)$ guarantees us the existence of $i_1 \in (X_k)_\lambda$, such that no seed fall on i_1 during $[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$. Since all the sites in $(X_k)_\lambda$ have been made vacant during the time interval $[\mathbf{a}_\lambda T_l, \mathbf{a}_\lambda(T_l + \varkappa_{\lambda, \pi})]$ (see Step 1), one easily concludes that i_1 is vacant during $[\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$.
- If $l = 0$ that is if $0 < T_q < 1$, there holds $0 < T_q < 1 - 2\alpha$ by $\Omega_M(\alpha)$. We conclude using $\Omega_2^S(\lambda, \pi)$ that there is a site $i_1 \in (X_k)_\lambda$ where no seed has fallen during $[0, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$ whence $\eta_{\mathbf{a}_\lambda s}^{\lambda, \pi}(i_1) = 0$ for all $s \in [\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$, as desired.

Case 3. Assume finally that $H_{T_q-}(x) = 0$ and that $T_q - [T_k - Z_{T_k-}(X_k)] \geq 1$, whence $T_q - [T_k - Z_{T_k-}(X_k)] \geq 1 + 2\alpha$ by $\Omega_M(\alpha)$. Since $H_{T_q-}(x) = 0$, there holds $Z_{T_q-}(c_-) < 1 = Z_{T_q-}(c_+)$ and $T_k + Z_{T_k-}(X_k) \leq T_q$, so that $T_k + Z_{T_k-}(X_k) \leq T_q - 2\alpha$.

We aim to use the event $\Omega_1^{S, P}(\lambda, \pi)$. We introduce

$$t_0 = T_k - Z_{T_k-}(X_k) = \tau_{T_k-}(X_k) = T_l.$$

Observe that $\tau_{T_k-}(c_-) = \tau_{T_k-}(c_+) = \tau_{T_k-}(x)$ because there has been no fire (exactly) at x during $[0, T_k)$. Thus $Z_{t_0-}(x) = Z_{t_0-}(x_-) = Z_{t_0-}(x_+) = 1$ and $Z_{t_0}(x) = Z_{t_0}(c_-) = Z_{t_0}(c_+) = 0$ (using the convention $Z_{0-}(y) = 1$ for all $y \in [-A, A]$).

Set now $t_1 = T_k$. Observe that $0 < t_1 - t_0 < 1$. Necessarily, $Z_t(c_-)$ has jumped to 0 at least one time between t_0 and T_q (else, one would have $Z_{T_q-}(c_-) = 1$, since $T_q - t_0 \geq 1$ by assumption) and this jump occurs after $t_0 + 1 > t_1$ (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$, and since for all $t \in [t_0, t_0 + 1)$, $Z_t(c_-) = t - t_0 < 1$).

We thus may denote by $t_2 < t_3 < \dots < t_K$, for some $K \geq 2$, the successive times of jumps of the process $(Z_t(c_-), Z_t(c_+))$ during $(t_0 + 1, T_q)$ and say x_2, \dots, x_K the corresponding locations of the fires. We also put $\varepsilon = 1$ if t_2 is a jump of $Z_t(c_+)$ and $\varepsilon = -1$ else.

Then we observe that $Z_t(c_-)$ and $Z_t(c_+)$ do never jump to 0 at the same time during (t_0, T_q) (else, it would mean that they are killed by the same fire at some time u , whence necessarily, $H_r(u) = 0$ and $Z_r(c_-) = Z_r(c_+)$ for all $r \in (u, T_q)$). Furthermore, there is always at least one jump of $(Z_t(c_-), Z_t(c_+))$ in any time interval of length 1 (during $[t_0 + 1, T_q)$), because else, $Z_t(c_+)$ and $Z_t(c_-)$ would both become equal to 1 and thus would remain equal forever. Finally, observe

that two jumps of $Z_t(c_-)$ cannot occur in a time interval of length 1 (since a jump of $Z_t(c_-)$ requires that $Z_t(c_-) = 1$) and the same thing holds for $Z_t(c_+)$.

Consequently, the family $\mathcal{P} = \{\varepsilon; (x_0, t_0), (X_k, T_k), \dots, (x_K, t_K)\}$ necessarily satisfies the condition (PP) of Subsection 9.3.

Next, there holds that $t_2 - t_1 < Z_{T_k-}(X_k) = t_1 - t_0$, because else, we would have $H_{t_2-}(X_k) = 0$ and thus the fire destroying c_+ (or c_-) at time t_2 would also destroy c_- (or c_+), we thus would have $Z_{t_2}(c_+) = Z_{t_2}(c_-) = 0$, so that $Z_t(c_+)$ and $Z_t(c_-)$ would remain equal forever. Furthermore, we have $t_K < T_q < t_K + 1$ because else, we would have $Z_{T_q-}(c_+) = Z_{T_q-}(c_-) = 1$.

Finally, we check that

$$(\eta_t^{\lambda, \pi}(i))_{t \in [\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi})], i \in (X_k)_\lambda} = (\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \in [\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi})], i \in (X_k)_\lambda},$$

this last process being built upon the families $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(N_t^P(i))_{t \geq 0, i \in \mathbb{Z}}$ as in Subsection 9.2. Indeed, seeds fall according to the same processes and fires propagate according to the same processes on $(X_k)_\lambda$. We already have checked that $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$ are equal on $(X_k)_\lambda$ during the time interval $[\mathbf{a}_\lambda t_0, \mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi})]$. Nothing happens on $(X_k)_\lambda$ during $[\mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda t_2]$. In both cases (say $\varepsilon = -1$), a match falls on $[\mathbf{n}_\lambda x_2] \in \llbracket -[\mathbf{n}_\lambda A], [\mathbf{n}_\lambda X_k] - \mathbf{m}_\lambda \rrbracket$ at time $\mathbf{a}_\lambda t_2$. This fire destroys the zone containing $[\mathbf{n}_\lambda X_k] - \mathbf{m}_\lambda$ (by definition of $\zeta^{\lambda, \pi, \mathcal{P}}$ and because, by construction, $D_{t_2-}(x_2) = [a, X_k]$, for some $a \in \mathcal{B}_M \cup \{-A\}$, whence $\eta_{\mathbf{a}_\lambda t_2-}^{\lambda, \pi}(j) = 1$ for all $j \in \llbracket [\mathbf{n}_\lambda x_2], [\mathbf{n}_\lambda X_m] - \mathbf{m}_\lambda \rrbracket$, see Steps 4 and 5 above) at the same time, since with our coupling, the second fire spreads according to the same rules and to the same processes in both cases. This implies that $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$ and $(\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$ are also equal on $(X_k)_\lambda$ during the time interval $[\mathbf{a}_\lambda(T_k + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_2 + \varkappa_{\lambda, \pi})]$. And so on.

We thus can use $\Omega_1^{S, P}(\lambda, \pi)$ and conclude that there is a site i_1 in $(X_k)_\lambda$ which is vacant during $[\mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_K + 1)]$ for $(\zeta_t^{\lambda, \pi, \mathcal{P}}(i))_{t \geq 0, i \in \mathbb{Z}}$. Since seeds fall on $(X_k)_\lambda$ according to the same processes, we deduce that there is also a vacant site in $(X_k)_\lambda$ during $[\mathbf{a}_\lambda(t_K + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda(t_K + 1)] \subset [\mathbf{a}_\lambda T_q, \mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})]$ for the (λ, π, A) -FFP, as desired.

Step 7. We now conclude. We put $z := Z_{T_q-}(X_q)$ and consider separately the cases $z \in (0, 1)$ and $z = 1$. Observe that $z = 0$ do never happens, since by construction, $Z_{T_q-}(X_q) = \min(Z_{T_{q-1}}(X_q) + T_q - T_{q-1}, 1) > 0$ and since $T_q > T_{q-1}$.

Case $z \in (0, 1)$. Then in the A -LFFP(0), we have $Z_{T_q-}(X_q) = Z_{T_q}(X_q)$ for all $x \in (-A, A)$ whence $\tau_{T_q-}(c) = \tau_{T_q}(c) = \tau_{T_q + \varkappa_{\lambda, \pi}}(c)$ for all $c \in \mathcal{C}_M$. Using Step 3, as seen in **Micro**(0) in Subsection 4.4, we see that the match falling on $[\mathbf{n}_\lambda X_q]$ at time $\mathbf{a}_\lambda T_q$ destroys nothing outside $\llbracket j_1, j_2 \rrbracket \subset (X_q)_\lambda$ and there is no more burning tree in I_A^λ at time $\mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi})$. We deduce that $\rho_s^{\lambda, \pi}(i) = \rho_{T_q}^{\lambda, \pi}(i)$ for all $s \in [T_q, T_q + \varkappa_{\lambda, \pi}]$ and all $i \notin (X_q)_\lambda$. Thus, applying $\Omega_{T_q}^{\lambda, \pi}$, we deduce that for all $c \in \mathcal{C}_M$ and all $i \in c_\lambda$,

$$\tau_{T_q + \varkappa_{\lambda, \pi}}(c) = \tau_{T_q}(c) \leq \rho_{T_q}^{\lambda, \pi}(i) = \rho_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}(i) \leq \tau_{T_q}(c) + \varkappa_{\lambda, \pi} = \tau_{T_q + \varkappa_{\lambda, \pi}}(c) + \varkappa_{\lambda, \pi}.$$

Thus, on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}$. Since no match falls on I_A^λ during $(\mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda T_{q+1})$ and since $\eta_{\mathbf{a}_\lambda T_{q+1}-}^{\lambda, \pi}(i) = \eta_{\mathbf{a}_\lambda T_{q+1}}^{\lambda, \pi}(i)$ for all $i \neq [\mathbf{n}_\lambda X_{q+1}]$, we deduce that on $\Omega(\alpha, \gamma, \lambda, \pi)$, for all $c \in \mathcal{C}_M$ and all $i \in c_\lambda$,

$$\rho_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}(i) = \rho_{T_{q+1}}^{\lambda, \pi}(i) \text{ and } \tau_{T_q + \varkappa_{\lambda, \pi}}(c) = \tau_{T_{q+1}}(c).$$

All this implies that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$ when $z \in (0, 1)$.

Case $z = 1$. Then there are $a, b \in \mathcal{B}_M \cup \{-A, A\}$ such that $D_{T_q-}(X_q) = [a, b]$. We assume that $a, b \in \mathcal{B}_M$, the other cases being treated similarly. By construction, we know that for all $c \in \mathcal{C}_M$ with $c \subset (a, b)$, $Z_{T_q-}(c) = 1$, for all $x \in \mathcal{B}_M \cap (a, b)$, $\tilde{H}_{T_q-}(x) = 0$ while finally $\tilde{H}_{T_q-}(a) > 0$ and $\tilde{H}_{T_q-}(b) > 0$.

For the A -LFFP(0), we have

- (i) $\tau_{T_q}(c) = T_q$ for all $c \in \mathcal{C}_M$ with $c \subset (a, b)$,
- (ii) $\tau_{T_q}(c) = \tau_{T_q-}(c)$ for all $c \in \mathcal{C}_M$ with $c \cap (a, b) = \emptyset$.

Next, using Steps 4, 5, using Step 6 for a (and a very similar result for b), we immediately check that the fire occurring on $[\mathbf{n}_\lambda X_q]$ at time $\mathbf{a}_\lambda T_q$, as seen in **Macro**(0) in Subsection 4.4,

- destroys completely all the cells $c \in \mathcal{C}_M$ with $c \subset (a, b)$,
- destroys completely all the zones $(x)_\lambda$ with $x \in \mathcal{B}_M \cap (a, b)$,
- does not destroy completely $(a)_\lambda$ nor $(b)_\lambda$,
- does not destroy at all the sites $i \in I_A^\lambda$ with $i \notin \llbracket [\mathbf{n}_\lambda a] - \mathbf{m}_\lambda, [\mathbf{n}_\lambda b] + \mathbf{m}_\lambda \rrbracket$.

Consequently, we have, for all $c \in \mathcal{C}_M$ with $c \subset (a, b)$ and all $i \in (c)_\lambda$,

$$\tau_{T_q + \varkappa_{\lambda, \pi}}(c) = \tau_{T_q}(c) = T_q \leq \rho_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}(i) \leq T_q + \varkappa_{\lambda, \pi} = \tau_{T_q}(c) + \varkappa_{\lambda, \pi} = \tau_{T_q + \varkappa_{\lambda, \pi}}(c) + \varkappa_{\lambda, \pi},$$

while if $c \cap (a, b) = \emptyset$, for all $i \in (c)_\lambda$,

$$\begin{aligned} \tau_{T_q + \varkappa_{\lambda, \pi}}(c) &= \tau_{T_q}(c) = \tau_{T_q-}(c) \leq \rho_{T_q-}^{\lambda, \pi}(i) = \rho_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}(i) \\ &\leq \tau_{T_q-}(c) + \varkappa_{\lambda, \pi} = \tau_{T_q}(c) + \varkappa_{\lambda, \pi} = \tau_{T_q + \varkappa_{\lambda, \pi}}(c) + \varkappa_{\lambda, \pi}. \end{aligned}$$

We conclude that when $z = 1$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}$. Since no match falls on I_A^λ during $[\mathbf{a}_\lambda(T_q + \varkappa_{\lambda, \pi}), \mathbf{a}_\lambda T_{q+1})$ and since $\eta_{\mathbf{a}_\lambda T_{q+1}-}^{\lambda, \pi}(i) = \eta_{\mathbf{a}_\lambda T_{q+1}}^{\lambda, \pi}(i)$ for all $i \neq [\mathbf{n}_\lambda X_{q+1}]$, we deduce that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q + \varkappa_{\lambda, \pi}}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$.

All this implies that on $\Omega(\alpha, \gamma, \lambda, \pi)$, $\Omega_{T_q}^{\lambda, \pi}$ implies $\Omega_{T_{q+1}}^{\lambda, \pi}$ when $z = 1$. This completes the proof. \square

9.5 Proof of Theorem 6.1 for $p = 0$

We finally give the proof of the Theorem 6.1 in the case $p = 0$. The proof is closely related to the proof in the case $p > 0$, recall Subsection 8.5.

Proof. Let us fix $x_0 \in (-A, A)$, $t_0 \in (0, T)$ and $\varepsilon > 0$. We will prove that with our coupling (see Subsection 9.4.1), when $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$, there holds that

- (a) $\lim_{\lambda, \pi} \mathbb{P} [\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) > \varepsilon] = 0;$
- (b) $\lim_{\lambda, \pi} \mathbb{P} [\delta_T(D^{\lambda, \pi}(x_0), D(x_0)) > \varepsilon] = 0;$
- (c) $\lim_{\lambda, \pi} \mathbb{P} [|Z_t^{\lambda, \pi}(x_0) - Z_t(x_0)| > \varepsilon] = 0;$
- (d) $\lim_{\lambda, \pi} \mathbb{P} \left[\int_0^T |Z_t^{\lambda, \pi}(x_0) - Z_t(x_0)| dt > \varepsilon \right] = 0;$
- (e) $\lim_{\lambda, \pi} \mathbb{P} [|W_{t_0}^{\lambda, \pi}(x_0) - Z_{t_0}(x_0)| > \varepsilon] = 0$, where

$$W_{t_0}^{\lambda, \pi}(x_0) = \left(\frac{\log(|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 1\}} \right) \wedge 1.$$

These points will clearly imply the result.

First, we introduce the event $\Omega_{A, T}^{x_0, t_0}(\alpha, \lambda, \pi)$ on which

- (i) $x_0 \notin \cup_{y \in \mathcal{B}_M} (y - 2\alpha, y + 2\alpha);$
- (ii) for all $s \in \mathcal{T}_M \cup \mathcal{S}_M$ with $s \leq t_0$, there holds that $t_0 - s > 2\alpha;$
- (iii) if $t_0 \neq 1$, for all $s \in \mathcal{T}_M \cup \mathcal{S}_M$ with $s \leq t_0$, there holds that $|t_0 - (s + 1)| > 2\alpha;$
- (iv) if $t_0 > 1$, for all $i \in I_A^\lambda$, $N_{\mathbf{a}_\lambda t_0}^S(i) - N_{\mathbf{a}_\lambda(t_0-1)}^S(i) > 0;$

(v) if $t_c = t_0 - \tau_{t_0-}(x_0) < 1$, there are i_1 and i_2 such that

$$-\lfloor \lambda^{-(t_c+\alpha)} \rfloor < i_1 < -\lfloor \lambda^{-(t_c-\alpha)} \rfloor < 0 < \lfloor \lambda^{-(t_c-\alpha)} \rfloor < i_2 < \lfloor \lambda^{-(t_c+\alpha)} \rfloor$$

and such that

- $N_{\mathbf{a}_\lambda t_0}^S(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) - N_{\mathbf{a}_\lambda \tau_{t_0-}(x_0)}^S(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_1) = 0$ whereas $N_{\mathbf{a}_\lambda t_0}^S(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) - N_{\mathbf{a}_\lambda \tau_{t_0-}(x_0)}^S(\lfloor \mathbf{n}_\lambda x_0 \rfloor + i_2) = 0$;
- for all $j \in \llbracket -\lfloor \lambda^{-(t_c-\alpha)} \rfloor, \lfloor \lambda^{-(t_c-\alpha)} \rfloor \rrbracket$, $N_{\mathbf{a}_\lambda t_0}^S(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) - N_{\mathbf{a}_\lambda(\tau_{t_0-}(x_0) + \varkappa_{\lambda,\pi})}^S(\lfloor \mathbf{n}_\lambda x_0 \rfloor + j) > 0$.

Since $t_0 - \tau_{t_0-}(x_0) = 1$ occurs with positive probability only if $t_0 = 1$ (and $\tau_{t_0-}(x_0) = 0$), the probability of the three first points clearly tend to 1 when α tends to 0. Since $(\tau_t(x_0))_{t \geq 0}$ is independent of $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$ and since $(\tau_t(x_0))_{t \geq 0} \subset \mathcal{T}_M \cup \mathcal{S}_M$, the probability of the two last points also tend to 1 as $\alpha \rightarrow 0$ and $\lambda \rightarrow 0$ and $\pi \rightarrow \infty$ in the regime $\mathcal{R}(0)$, thanks to Lemma 9.1-4,6,7 and space/time stationarity (recall that $\varkappa_{\lambda,\pi} \rightarrow 0$). All this implies that for all $\delta > 0$, there is $\alpha > 0$ such that $\mathbb{P} \left[\Omega_{A,T}^{x_0,t_0}(\alpha, \lambda, \pi) \right] > 1 - \delta$ for all (λ, π) sufficiently close to the regime $\mathcal{R}(0)$.

Let us now fix $\delta > 0$. In the rest of the proof, we consider $\alpha_0 \in (0, \varepsilon)$, $\gamma_0 \in (0, \alpha_0)$, $\lambda_0 \in (0, 1)$ and $\epsilon_0 \in (0, 1)$ such that for all $\lambda \in (0, \lambda_0)$ and all $\pi \geq 1$ in such a way that $\mathbf{n}_\lambda / (\mathbf{a}_\lambda \pi) < \epsilon_0$, we have

$$\mathbb{P} \left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0,t_0}(\alpha_0, \lambda, \pi) \right] > 1 - \delta.$$

We then consider $\lambda_1 \in (0, \lambda_0)$ and $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\lambda \in (0, \lambda_1)$ and all $\pi \geq 1$ in such a way that $\mathbf{n}_\lambda / (\mathbf{a}_\lambda \pi) < \epsilon_1$, we have

- $\varkappa_{\lambda,\pi} \leq \alpha_0$;
- $\alpha_0 + \log(\mathbf{a}_\lambda) / \log(1/\lambda) < \varepsilon$;
- $4\mathbf{m}_\lambda / \mathbf{n}_\lambda \leq \varepsilon$;
- $1/(2\mathbf{m}_\lambda \lambda^{t_c - 2\varepsilon}) \leq \delta$ and $1/(2\mathbf{m}_\lambda \lambda^{t_c + \varkappa_{\lambda,\pi}}) \leq \delta$ if $t_c < 1$.

All this can be done properly by using the fact that $\varkappa_{\lambda,\pi} \rightarrow 0$ and $\mathbf{m}_\lambda / \mathbf{n}_\lambda \rightarrow 0$.

In the rest of the proof, we consider $\lambda \in (0, \lambda_1)$ and $\pi \geq 1$ in such a way that $\mathbf{n}_\lambda / (\mathbf{a}_\lambda \pi) \leq \epsilon_1$. Observe that, on $\Omega_{A,T}^{x_0,t_0}(\alpha_0, \lambda, \pi)$, we have $\tau_{t_0-}(x_0) = \tau_{t_0}(x_0)$ and $(x_0)_\lambda \cap (\bigcup_{x \in \mathcal{B}_M} (x)_\lambda) = \emptyset$. We call $c_0 \in \mathcal{C}_M$ the cell containing x_0 .

Step 1. As in Subsection 8.5, Steps 1 and 2, (a) (which holds for an arbitrary value of $t_0 \in (0, T)$) implies (b) and (c) implies (d).

Step 2. Due to Lemma 9.4, we know that, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0,t_0}(\alpha_0, \lambda, \pi)$, since $t_0 > \tau_{t_0}(x_0) + 3\alpha_0$, for all $i \in (x_0)_\lambda$,

$$\tau_{t_0}(c_0) \leq \rho_{t_0}^{\lambda,\pi}(i) \leq \tau_{t_0}(c_0) + \varkappa_{\lambda,\pi}.$$

For all $i \in (x_0)_\lambda$, since $\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) \leq 1$, there holds

$$\eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) = \min(N_{\mathbf{a}_\lambda t_0}^{S,\lambda,\pi}(i) - N_{\mathbf{a}_\lambda \rho_{t_0}^{\lambda,\pi}(i)}^{S,\lambda,\pi}(i), 1).$$

Thus, for all $i \in (x_0)_\lambda$,

$$\underline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) \leq \eta_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) \leq \overline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i)$$

where

$$\begin{aligned} \underline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) &:= \min(N_{\mathbf{a}_\lambda t_0}^S(i) - N_{\mathbf{a}_\lambda(\tau_{t_0}(x_0) + \varkappa_{\lambda,\pi})}^S(i), 1), \\ \overline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda,\pi}(i) &:= \min(N_{\mathbf{a}_\lambda t_0}^S(i) - N_{\mathbf{a}_\lambda \tau_{t_0-}(x_0)}^S(i), 1). \end{aligned}$$

We also recall that by construction, $(\tau_t(x_0))_{t \geq 0}$ is independent of $(N_t^S(i))_{t \geq 0, i \in \mathbb{Z}}$.

Step 3. Here we prove (e). We work on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$. By Step 2 and point (v) of the event $\Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we observe that if $0 < t_c = t_0 - \tau_{t_0}(x_0) < 1$, then

$$\begin{aligned} & \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor - \lfloor \lambda^{-(t_c - \alpha_0)} \rfloor, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \lfloor \lambda^{-(t_c - \alpha_0)} \rfloor \rrbracket \\ & \subset C(\underline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset C(\overline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \\ & \subset \llbracket \lfloor \mathbf{n}_\lambda x_0 \rfloor - \lfloor \lambda^{-(t_c + \alpha)} \rfloor, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \lfloor \lambda^{-(t_c + \alpha)} \rfloor \rrbracket. \end{aligned}$$

Thus, this implies that

$$|W_{t_0}^{\lambda, \pi}(x_0) - (t_0 - \tau_{t_0}(x_0))| \leq \alpha_0 + \frac{\log(2)}{\log(1/\lambda)} < \varepsilon.$$

If now $t_0 - \tau_{t_0}(x_0) > 1$, then $t_0 - \tau_{t_0}(x_0) > 1 + 2\alpha_0$ thanks to $\Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$. Then Step 2 and point (iv) of $\Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$ imply that $(x_0)_\lambda \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)$ whence $|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 2\mathbf{m}_\lambda$. Consequently,

$$W_{t_0}^{\lambda, \pi}(x_0) \geq 1 - \frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)} > 1 - \varepsilon.$$

It only remains to study what happens when $t_0 = 1$. By construction, we have $\tau_{t_0}(x_0) = 0$. Observe that on $\Omega(\alpha, \gamma, \lambda, \pi)$, a match falling on $\lfloor \mathbf{n}_\lambda X_k \rfloor$ at time $\mathbf{a}_\lambda T_k \leq 1$, for some $k \in \{1, \dots, n\}$, does not affect the zone outside $(X_k)_\lambda$. Thus, for all $i \in (x_0)_\lambda$,

$$\eta_{\mathbf{a}_\lambda}^{\lambda, \pi}(i) = \min(N_{\mathbf{a}_\lambda}^S(i), 1).$$

Using point (iv) of the event $\Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we deduce that

$$(x_0)_\lambda \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)$$

and conclude that $|C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor)| \geq 2\mathbf{m}_\lambda$, whence

$$W_{t_0}^{\lambda, \pi}(x_0) \geq 1 - \frac{\log(\mathbf{a}_\lambda)}{\log(1/\lambda)} \geq 1 - \varepsilon.$$

Recalling that $Z_{t_0}(x_0) = (t_0 - \tau_{t_0}(x_0)) \wedge 1$, we have proved that

$$\mathbb{P} \left[|W_{t_0}^{\lambda, \pi}(x_0) - Z_{t_0}(x_0)| < \varepsilon \right] \geq \mathbb{P} \left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi) \right] \geq 1 - \delta,$$

as desired.

Step 4. Here we prove (c). Recall that $Z_{t_0}^{\lambda, \pi}(x_0) = \left(-\frac{\log(1 - K_{t_0}^{\lambda, \pi}(x_0))}{\log(1/\lambda)} \right) \wedge 1$ where $K_{t_0}^{\lambda, \pi}(x_0) = (2\mathbf{m}_\lambda + 1)^{-1} \left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda X_0 \rfloor + \mathbf{m}_\lambda \rrbracket : \eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1 \right\} \right|$. We work on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A,T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$ and set $t_c = t_0 - \tau_{t_0}(x_0)$.

Case 1. If $t_c \geq 1$, we have checked in Step 3 that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1$ for all $i \in (x_0)_\lambda$, whence $K_{t_0}^{\lambda, \pi}(x_0) = 1$ and $Z_{t_0}^{\lambda, \pi}(x_0) = 1$.

Case 2. If now $0 < t_c < 1$, we deduce from Step 3 that

$$\underline{K}_{t_0}^{\lambda, \pi}(x_0) \leq K_{t_0}^{\lambda, \pi}(x_0) \leq \overline{K}_{t_0}^{\lambda, \pi}(x_0)$$

where

$$\begin{aligned} \underline{K}_{t_0}^{\lambda, \pi}(x_0) &= (2\mathbf{m}_\lambda + 1)^{-1} \left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda \rrbracket : \underline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1 \right\} \right|, \\ \overline{K}_{t_0}^{\lambda, \pi}(x_0) &= (2\mathbf{m}_\lambda + 1)^{-1} \left| \left\{ i \in \llbracket \lfloor \mathbf{n}_\lambda X_0 \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda x_0 \rfloor + \mathbf{m}_\lambda \rrbracket : \overline{\eta}_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1 \right\} \right|. \end{aligned}$$

Recalling Step 5 in Subsection 8.5, we deduce that

$$\mathbb{P} \left[K_{t_0}^{\lambda, \pi}(x_0) \in (1 - \lambda^{t_c - \varepsilon}, 1 - \lambda^{t_c + \varepsilon}) \right] \geq 1 - c\delta,$$

for some constant $c > 0$, whence

$$\mathbb{P} \left[Z_{t_0}^{\lambda, \pi}(x_0) \in (t_c - \varepsilon, t_c + \varepsilon) \right] \geq 1 - c\delta.$$

This is nothing but the goal, since $Z_{t_0}(x_0) = t_0 - \tau_{t_0}(x_0) = t_c$ as soon as $Z_{t_0}(x_0) < 1$.

Step 5. It remains to prove (a). On $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we check that

- (i) If $Z_{t_0}(x_0) < 1$, then $D_{t_0}(x_0) = \{x_0\}$ and $C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset (x_0)_\lambda$ (see Step 3 above), whence $D_{t_0}^{\lambda, \pi}(x_0) \subset [x_0 - \mathbf{m}_\lambda / \mathbf{n}_\lambda, x_0 + \mathbf{m}_\lambda / \mathbf{n}_\lambda]$. We deduce that $\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq 2\mathbf{m}_\lambda / \mathbf{n}_\lambda$.
- (ii) If $Z_{t_0}(x_0) = 1$ and $D_{t_0}(x_0) = [a, b]$, for some $a, b \in \mathcal{B}_M \cup \{-A, A\}$, then
 - for $c \in \mathcal{C}_M$ with $c \subset (a, b)$, $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1$ for all $i \in c_\lambda$ (see Step 4 of the preceding proof);
 - for $x \in \mathcal{B}_M \cap (a, b)$, $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = 1$ for all $i \in (x)_\lambda$ (see Step 5 of the preceding proof);
 - there are $i \in (a)_\lambda$ and $j \in (b)_\lambda$ such that $\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(i) = \eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}(j) = 0$ (see Step 6 of the preceding proof);

so that

$$\llbracket \lfloor \mathbf{n}_\lambda a \rfloor + \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda b \rfloor - \mathbf{m}_\lambda \rrbracket \subset C(\eta_{\mathbf{a}_\lambda t_0}^{\lambda, \pi}, \lfloor \mathbf{n}_\lambda x_0 \rfloor) \subset \llbracket \lfloor \mathbf{n}_\lambda a \rfloor - \mathbf{m}_\lambda, \lfloor \mathbf{n}_\lambda b \rfloor + \mathbf{m}_\lambda \rrbracket$$

and thus

$$[a + \mathbf{m}_\lambda / \mathbf{n}_\lambda, b - \mathbf{m}_\lambda / \mathbf{n}_\lambda] \subset D_{t_0}^{\lambda, \pi}(x_0) \subset [a - \mathbf{m}_\lambda / \mathbf{n}_\lambda, b + \mathbf{m}_\lambda / \mathbf{n}_\lambda],$$

whence $\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq 4\mathbf{m}_\lambda / \mathbf{n}_\lambda$.

Thus, on $\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi)$, we always have $\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq 4\mathbf{m}_\lambda / \mathbf{n}_\lambda$. We conclude that

$$\mathbb{P} \left[\delta(D_{t_0}^{\lambda, \pi}(x_0), D_{t_0}(x_0)) \leq \varepsilon \right] \geq \mathbb{P} \left[\Omega(\alpha_0, \gamma_0, \lambda, \pi) \cap \Omega_{A, T}^{x_0, t_0}(\alpha_0, \lambda, \pi) \right] \geq 1 - \delta.$$

This concludes the proof. \square

9.6 Cluster size distribution when $p = 0$

The aim of this section is to prove Corollary 2.7 when $p = 0$. We first recall a result of [[6], Lemma 3.11.1].

Lemma 9.5. *Let $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(0) and consider $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$ the associated process. There are some constants $0 < c_1 < c_2$ and $0 < \kappa_1 < \kappa_2$ such that the following estimates hold.*

- (i) For any $t \in (1, \infty)$, any $x \in \mathbb{R}$, any $z \in [0, 1)$, $\mathbb{P}[Z_t(x) = z] = 0$.
- (ii) For any $t \in [0, \infty)$, any $B > 0$, any $x \in \mathbb{R}$, $\mathbb{P}[|D_t(x)| = B] = 0$.
- (iii) For all $t \in [0, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \leq c_2 e^{-\kappa_1 B}$.
- (iv) For all $t \in [\frac{3}{2}, \infty)$, all $x \in \mathbb{R}$, all $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \geq c_1 e^{-\kappa_2 B}$.
- (v) For all $t \in [5/2, \infty)$, all $0 \leq a < b < 1$, all $x \in \mathbb{R}$,

$$c_1(b - a) \leq \mathbb{P}[Z_t(x) \in [a, b]] \leq c_2(b - a).$$

We now handle the

Proof of Corollary 2.7 when $p = 0$. For each $\lambda \in (0, 1)$ and each $\pi \geq 1$, consider a (λ, π) -FFP $(\eta_t^{\lambda, \pi}(i))_{t \geq 0, i \in \mathbb{Z}}$. Let also $(Z_t(x), H_t(x), F_t(x))_{t \geq 0, x \in \mathbb{R}}$ be a LFFP(0) and consider the corresponding process $(D_t(x))_{t \geq 0, x \in \mathbb{R}}$.

Point (b). Using Lemma 9.5-(iii)-(iv) and recalling that $|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)|/\mathbf{n}_\lambda = |D_t^{\lambda, \pi}(0)|$, it suffices to check that for all $t \geq 3/2$ and all $B > 0$, when $\lambda \rightarrow 0$ and $\pi \rightarrow 0$ in the regime $\mathcal{R}(0)$,

$$\lim_{\lambda, \pi} \mathbb{P} \left[|D_t^{\lambda, \pi}(0)| \geq B \right] = \mathbb{P} [|D_t(0)| \geq B].$$

This follows from Theorem 2.5-2, which implies that $|D_t^{\lambda, \pi}(0)|$ goes in law to $|D_t(0)|$ and from Lemma 9.5-(ii).

Point (a). Due to Lemma 9.5-(v) we only need that for all $0 < a < b < 1$, all $t \geq 5/2$, when $\lambda \rightarrow 0$ and $\pi \rightarrow 0$ in the regime $\mathcal{R}(0)$,

$$\lim_{\lambda, \pi} \mathbb{P} \left[|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)| \in [\lambda^{-a}, \lambda^{-b}] \right] = \mathbb{P} [Z_t(0) \in [a, b]].$$

But using Theorem 2.5-3 and Lemma 9.5-(i), we know that

$$\lim_{\lambda, \pi} \mathbb{P} \left[\frac{\log(|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)|)}{\log(1/\lambda)} \mathbf{1}_{\{|C(\eta_{\mathbf{a}_\lambda t}^{\lambda, \pi}, 0)| \geq 1\}} \in [a, b] \right] = \mathbb{P} [Z_t(0) \in [a, b]]$$

as $\lambda \rightarrow 0$ and $\pi \rightarrow 0$ in the regime $\mathcal{R}(0)$. One immediately concludes. \square

Bibliography

- [1] P. Bak, C. Tang, K. Wiesenfeld, *Self-organized criticality: an explanation of $1/f$ noise*, Phys. Rev. Letters **59** (1987), 381–384.
- [2] P. Bak, C. Tang, K. Wiesenfeld, *Self-organized criticality*, Phys. Rev. A **38** (1988), 364–374.
- [3] J. van den Berg, R. Brouwer, *Self-organized forest-fires near the critical time*, Comm. Math. Phys. **267** (2006), 265–277.
- [4] J. van den Berg, A.A. Járai, *On the asymptotic density in a one-dimensional self-organized critical forest-fire model*, Comm. Math. Phys. **253** (2005), 633–644.
- [5] X. Bressaud, N. Fournier, *Asymptotics of one-dimensional forest fire processes.*, Ann. Probab. **38** (2010), 1783–1816.
- [6] X. Bressaud, N. Fournier, *One-dimensional general forest fire processes.*, Mém. Soc. Math. Fr. **132** (2013), vi+138 pages
- [7] R. Brouwer and J. Pennanen, *The cluster size distribution for a forest-fire process on \mathbb{Z}* , Electron. J. Probab. **11** (2006), 1133–1143.
- [8] B. Drossel, S. Clar, F. Schwabl, *Exact results for the one-dimensional self-organized critical forest-fire model*, Phys. Rev. Lett. **71** (1993), 3739–3742.
- [9] B. Drossel, F. Schwabl, *Self-organized critical forest-fire model*, Phys. Rev. Lett. **69** (1992), 1629–1632.
- [10] M. Dürre, *Existence of multi-dimensional infinite volume self-organized critical forest-fire models*, Electron. J. Probab. **11** (2006), 513–539.
- [11] M. Dürre, *Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models*, Electronic Communications in Probability **11** (2006), 304–315.
- [12] M. Dürre, *Self-organized critical phenomena: Forest fire and sand pile models*, PhD Dissertation, LMU München, 2009.
- [13] P. Grassberger, *Critical Behaviour of the Drossel-Schwabl Forest Fire Model*, New Journal of Physics **4** (2002), 17.
- [14] C.L. Henley, *Self-organized percolation: a simpler model*, Bull. Am. Phys. Soc. **34** (1989), 838.
- [15] T.M. Liggett, *Interacting particle systems*, Springer, 1985.
- [16] Z. Olami, H. J. S. Feder, K. Christensen, *Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes*, Physical Review Letters **68** (1992), 1244–1247.